Math 316 (Fund. of analysis), Winter 2018
Quiz 1 Solutions

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Date: 2/2/2018

Name:

Q1		6
Q2		6
Q3		8
Q4		20
Q5		10
Total		50

General notes:

1. Terminology:
   - Supremum = least upper bound.
   - Injective = 1-to-1 = into.
   - Surjective = onto.
2. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (**one or two sentences**).
   - To justify that "Every closed interval contains 0" is false, you should provide a counterexample, such as $[1, 2]$.
   - To justify that "Some closed interval contains 0" is true, you should provide an example, such as $[-1, 1]$.
   - To justify that "Every closed interval contains some nonzero number" is true, you should explain. "For any interval $[a, b]$, either $a \neq 0$ or $b \neq 0$."
3. If I want a complete proof, I will say "Prove that ..."
4. If I don't need any justification, I will say "no justification necessary." No justification is needed for a definition.

1. **(6 pts)** Let A be a subset of $\mathbb{R}$. Give a formal definition of the *infimum* or *greatest lower bound* of A.

   A real number $x \in \mathbb{R}$ is an *infimum* of A if

   - x is a lower bound for A (i.e. $x \leq a$ for all $a \in A$), and
   - for any lower bound y for A, one has $y \leq x$.

2. **(6 pts)** Consider the set

   $$B = \{3 - \frac{1}{n} \mid n \in \mathbb{N}\} \cup (497, 500)$$

   inside $\mathbb{R}$. Does B have more than one supremum? Compute every supremum of B (no justification necessary).

   Every set has at most one supremum. $\sup B = 500$.

3. **(8 pts)** Suppose C and D are nonempty subsets of $\mathbb{R}$ which are bounded above, but not bounded below. Define the set

   $$C \cdot D = \{c \cdot d \mid c \in C, d \in D\}.$$

   Does $\sup(C \cdot D)$ exist? If so, find a formula for it. If not, find a counterexample.

   $\sup(C \cdot D)$ does not exist, because $C \cdot D$ is not bounded above. For example, $C = D = (-\infty, 0)$. Then $C \cdot D = (0, \infty)$.

4. **(20 pts)** For each of the following statements, is it true or false? Justification is required.
   (a) Any open interval in $\mathbb{R}$ contains an irrational number.
   (b) Every nonempty subset of $\mathbb{R}$ has a supremum.
   (c) There is no collection of open intervals I_n such that $\bigcap_{n\in\mathbb{N}} I_n$ is a closed interval.
   (d) The set {even numbers} $\cup$ {multiples of 7} is countable.

a) True. The irrational numbers are dense. (This would be enough of an answer. If you want more:) This can be seen because the numbers of the form $\mathbb{Q}+\sqrt{2}$ are dense.

b) False. The subset must be bounded. Counterexample: $\mathbb{N}$.

c) False. Counterexample: $I_n = (4-\frac{1}{n}, 5+e^{-n})$, then $\bigcap_{n\in\mathbb{N}} I_n = [4,5]$.

d) True. (Two good answers. First:) This is a subset of $\mathbb{N}$, and any subset of a countable set is countable. (Second:) Clearly $2\mathbb{N}$ and $7\mathbb{N}$ are in bijection with $\mathbb{N}$. A finite union of countable sets is countable.

5. **(10 pts)** Let $a < b$ be real numbers, and consider the set $T = \mathbb{Q}\cap[a,b]$. Prove that $\sup T = b$.

Note that b is an upper bound for $[a,b]$, so it is an upper bound for T. It remains to show that if c is an upper bound for T, then $b \le c$. So suppose instead that $c < b$.

First observe that, by density of $\mathbb{Q}$, there is a rational number $r \in \mathbb{Q}\cap[a,b]$. Since $r \le c$ and $a \le r$, we have $a \le c$. Now, again use density of $\mathbb{Q}$ to find a rational number $t \in \mathbb{Q}\cap(c,b)$. Then $a \le c < t < b$ so $t \in \mathbb{Q}\cap[a,b] = T$. But $c < t$, so c is not an upper bound for T, a contradiction.

Thus $b \le c$ for any other upper bound for T, and b is the supremum.