Math 316 (Fund. of analysis), Winter 2018
Midterm 1
Teacher: Ben Elias
Date: 2/12/2018

Name:

Q1		10
Q2		10
Q3		12
Q4		10
Q5		36
Q6		10
Q7		12
Total		100

General notes:

1. Terminology:
   - Supremum = least upper bound.
   - Injective = 1-to-1 = into.
   - Surjective = onto.
2. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (**one or two sentences**).
   - To justify that "Every closed interval contains 0" is false, you should provide a counterexample, such as $[1, 2]$.
   - To justify that "Some closed interval contains 0" is true, you should provide an example, such as $[-1, 1]$.
   - To justify that "Every closed interval contains some nonzero number" is true, you should explain. "For any interval $[a, b]$, either $a \neq 0$ or $b \neq 0$."
3. If I want a complete proof, I will say "Prove that ..."
4. If I don't need any justification, I will say "no justification necessary." No justification is needed for a definition.
5. You can use any theorem from the book, from class, or from homework, unless specifically stated otherwise.

1. **(10 pts, 5 each)** Give formal definitions for the following concepts.

   (a) What it means for the real numbers to be *complete*.

   (b) What it means for a sequence (x_n) of real numbers to *diverge*.

2. **(10 pts, 5 each)**

   (a) Let C_1 and C_2 be nonempty subsets of $\mathbb{R}$ which are each bounded above. Does $\sup(C_1 \cup C_2)$ exist? If so, find a formula for it (no justification required). If not, find a counterexample.

   (b) Now let $C_1, C_2, C_3, \ldots$ be nonempty subsets of $\mathbb{R}$ which are each bounded above. Does $\sup(\bigcup_{n \in \mathbb{N}} C_n)$ exist? If so, find a formula for it (no justification required). If not, find a counterexample.

3. **(12 pts, 3 each)** Suppose that (x_n) converges to -12. For each of the statements below, is it true or false? (No justification needed. No partial credit.)

   (a) Only finitely many of the terms x_n can be positive.

   (b) At least one term x_n is less than -11.5.

   (c) If $x_N = -12$ for some N, then $x_n = -12$ for all $n \geq N$.

   (d) None of the terms x_n can be less than -1000.

4. **(10 pts)** Let X be a countably infinite subset of $\mathbb{R}$. Fix an element $y \in \mathbb{R}$ with $y \notin X$. Prove that $X \cup \{y\}$ is countably infinite, using only the definition of countability (e.g. do not use the fact that a union of countable sets is countable).

5. **(36 pts, 6 each)** For each of the following statements, is it true or false? Justification is required.

   (a) Any open interval in $\mathbb{R}$ contains a rational number of the form $\frac{p}{q}$ for $q \leq 9999$.

   (b) (Let $\mathbf{I} = \mathbb{R} \setminus \mathbb{Q}$ denote the irrational numbers.) For any infinite subset $X \subset \mathbf{I}$, X is uncountable.

   (c) If $\lim x_n = x$ and $\lim y_n = y \neq 0$, then $\lim \left(\frac{x_n+3}{y_n^2} \right) = \frac{x+3}{y^2}$.

   (d) Every increasing sequence of negative numbers has a limit.

   (e) If a sequence does not converge to 5, then infinitely many terms of the sequence must be outside the interval $(4, 6)$.

   (f) Let $y_n = |x_n - x_{n+1}|$. If y_n converges to zero, then x_n converges.

6. **(10 pts)** Prove that
$$\lim \frac{7n+2}{2n-5} = \frac{7}{2}.$$

7. **(12 pts)** If (x_n) converges to 101, and $x_n \neq 0$ for all $n \in \mathbb{N}$, then $\frac{1}{x_n}$ converges to $\frac{1}{101}$. Prove this directly from the definition of convergence (that is, without using the algebraic limit theorem).