# Math 316 (Fund. of analysis), Winter 2018
## Midterm 1

Teacher: Ben Elias
Date: 2/12/2018

Name:

| | | |
|---|---|---|
| Q1 | | 10 |
| Q2 | | 10 |
| Q3 | | 12 |
| Q4 | | 10 |
| Q5 | | 36 |
| Q6 | | 10 |
| Q7 | | 12 |
| Total | | 100 |

**General notes:**

1. Terminology:
   - Supremum = least upper bound.
   - Injective = 1-to-1 = into.
   - Surjective = onto.
2. Justification: On an exam, by default, you should justify your statements. If an example or counterexample is the best justification, you should provide one. Otherwise, you should provide a brief conceptual explanation (**one or two sentences**).
   - To justify that "Every closed interval contains 0" is false, you should provide a counterexample, such as $[1, 2]$.
   - To justify that "Some closed interval contains 0" is true, you should provide an example, such as $[-1, 1]$.
   - To justify that "Every closed interval contains some nonzero number" is true, you should explain. "For any interval $[a, b]$, either $a \neq 0$ or $b \neq 0$."
3. If I want a complete proof, I will say "Prove that ..."
4. If I don't need any justification, I will say "no justification necessary." No justification is needed for a definition.
5. You can use any theorem from the book, from class, or from homework, unless specifically stated otherwise.

1. **(10 pts, 5 each)** Give formal definitions for the following concepts.

   (a) What it means for the real numbers to be *complete*.

   (b) What it means for a sequence $(x_n)$ of real numbers to *diverge*.

(a) The axiom of completeness states that any nonempty subset $X \subset \mathbb{R}$ which is bounded above has a supremum in $\mathbb{R}$.
Common mistakes:

- Confusing completeness with density. Remember, completeness is the thing which makes the real numbers tick - it is the one property that differentiates between $\mathbb{R}$ and $\mathbb{Q}$. But $\mathbb{Q}$ is dense, so that can't be completeness.
- Also, showing that, for all $a, b \in \mathbb{R}$ with $a < b$, there is a real number $r$ with $a < r < b$... this is easy. Let $r = \frac{a+b}{2}$. Then $r$ works. It would be very silly if I spent two weeks talking about that.
- Forgetting the word "nonempty." Does the empty set have a supremum? Does it have any upper bounds?
- Forgetting the condition that $X$ is bounded above. Of course, this is really important.

A sequence $(x_n)$ diverges if $x_n$ does not converge to $a$ for any real number $a$.
Common mistakes:

- Writing down the negation of the definition of convergence to $a$, without ever saying what $a$ is!
- Not having a clause "for any real number $a$", or something equivalent to that. Without that, one is merely making a terminological point - diverge is the opposite of converge - while not really defining either. But "convergence to $a$" is a definite thing, and this can be referred to.
- Writing down something involving the definition of convergence, but getting the quantifiers wrong.
- Writing down "$(x_n)$ diverges if it is not Cauchy." This is not a definition, this is a theorem.

2. **(10 pts, 5 each)**

   (a) Let $C_1$ and $C_2$ be nonempty subsets of $\mathbb{R}$ which are each bounded above. Does $\sup(C_1 \cup C_2)$ exist? If so, find a formula for it (no justification required). If not, find a counterexample.

   (b) Now let $C_1, C_2, C_3, \ldots$ be nonempty subsets of $\mathbb{R}$ which are each bounded above. Does $\sup(\bigcup_{n\in\mathbb{N}} C_n)$ exist? If so, find a formula for it (no justification required). If not, find a counterexample.

a) Yes, and $\sup(C_1 \cup C_2) = \max\{\sup C_1, \sup C_2\}$.

The problem does not ask for a proof, but here is one: $C_1 \cup C_2$ is nonempty, because $C_1$ is. If $M_1$ and $M_2$ are upper bounds for $C_1$ and $C_2$ respectively, then $\max\{M_1, M_2\}$ is an upper bound for $C_1 \cup C_2$, as is easily shown. Let $s = \max\{\sup C_1, \sup C_2\}$. Then $s$ is an upper bound for $C_1 \cup C_2$. WLOG let $s = \sup C_1 \geq \sup C_2$. If $t < s$ then $t < \sup C_1$ so $t$ is not an upper bound for $C_1$, so there is some $c \in C_1$ with $t < c$. Then $c \in C_1 \cup C_2$, so $t$ is not an upper bound for $C_1 \cup C_2$. Thus $s$ is the lowest upper bound.

b) It need not exist, since $\bigcup_{n\in\mathbb{N}} C_n$ need not be bounded above. For example, let $C_i = \{i\}$, the one point set. Then $\bigcup C_n = \mathbb{N}$.
Common mistakes:

- Writing down something like $\sup C_1 \cup \sup C_2$. $\sup C_1$ is a number, not a set. You can't take the union of two numbers.
- Assuming that the infinite union in (b) is bounded above.

3. **(12 pts, 3 each)** Suppose that $(x_n)$ converges to $-12$. For each of the statements below, is it true or false? (No justification needed. No partial credit.)
   (a) Only finitely many of the terms $x_n$ can be positive.
   (b) At least one term $x_n$ is less than $-11.5$.
   (c) If $x_N = -12$ for some $N$, then $x_n = -12$ for all $n \geq N$.
   (d) None of the terms $x_n$ can be less than $-1000$.

a) True. Letting $\epsilon = 2$ (as a random example), eventually $x_n < -12 + 2 = -10 < 0$. Only finitely many terms can violate this bound.

b) True. Letting $\epsilon = .5$, eventually $x_n < -12 + .5 = -11.5$.

c) False. The sequence $x_1 = -12$, $x_n = -11 - \frac{1}{n}$ for $n \geq 2$ converges to $-12$.

d) False. The sequence $x_1 = -2000$, $x_n = -11 - \frac{1}{n}$ for $n \geq 2$ converges to $-12$.

Word of advice - when thinking about a sequence, and what can be true, always remember this: if you change the first 1000000 terms of the sequence to be arbitrary crazy numbers, it does not change whether the sequence converges, or what its limit is. If you understand this, you can not get (c) or (d) wrong.

4. **(10 pts)** Let $X$ be a countably infinite subset of $\mathbb{R}$. Fix an element $y \in \mathbb{R}$ with $y \notin X$. Prove that $X \cup \{y\}$ is countably infinite, using only the definition of countability (e.g. do not use the fact that a union of countable sets is countable).

By definition of countable, there is a bijective function $f \colon \mathbb{N} \to X$. We must define a bijective function $g \colon \mathbb{N} \to X \cup \{y\}$. Let $g(1) = y$ and $g(n) = f(n-1)$ for $n \geq 2$. This works, as confirmed below.

$g$ is injective because: $g(1) \neq g(n)$ for any $n \neq 1$, since $y \notin X$ but $f(n-1) \in X$; and $g(n) \neq g(m)$ for any $n \neq m$, both $\neq 1$, since $f$ is injective.

$g$ is surjective because: $y = g(1)$; and for any $x \in X$, there exists $n \in \mathbb{N}$ with $f(n) = x$, and thus $g(n+1) = x$.

So many varied mistakes here, mostly stemming from not knowing the definition of countability. The proof tells you to use the definition, so use it. That means you MUST construct a bijection from $\mathbb{N}$ to $X \cup \{y\}$.

5. **(36 pts, 6 each)** For each of the following statements, is it true or false? Justification is required.

   (a) Any open interval in $\mathbb{R}$ contains a rational number of the form $\frac{p}{q}$ for $q \leq 9999$.

   (b) (Let $\mathbf{I} = \mathbb{R} \setminus \mathbb{Q}$ denote the irrational numbers.) For any infinite subset $X \subset \mathbf{I}$, $X$ is uncountable.

   (c) If $\lim x_n = x$ and $\lim y_n = y \neq 0$, then $\lim \left( \frac{x_n+3}{y_n^2} \right) = \frac{x+3}{y^2}$.

   (d) Every increasing sequence of negative numbers has a limit.

   (e) If a sequence does not converge to 5, then infinitely many terms of the sequence must be outside the interval $(4, 6)$.

   (f) Let $y_n = |x_n - x_{n+1}|$. If $y_n$ converges to zero, then $x_n$ converges.

a) False. The interval $(0, \frac{1}{10000})$ contains no such rational numbers.

(No one actually tried to justify that this interval has no rational numbers with $q \leq 9999$, but here's the super quick proof. If

$$0 < \frac{p}{q} < \frac{1}{10000}$$

for $q > 0$, then $q > 10000p$ and $p \geq 1$, meaning that $q > 10000$.)

Common mistakes:

- The closest answers to correct wrote $(\frac{1}{10000}, \frac{2}{10000})$. But this interval has tons of rational numbers with $q \leq 9999$, starting with $\frac{1}{5001}$. Right idea though - find a very narrow interval.
- Density of $\mathbb{Q}$ just doesn't help at all. And the whole idea behind the proof of the density of $\mathbb{Q}$ is that, by making $q$ arbitrarily large, you can squeeze into any small interval. Bounding $q$ destroys this proof.

b) False. The set $\{n + \pi \mid n \in \mathbb{N}\}$ is in obvious bijection with $\mathbb{N}$.

Common mistakes:

- A subset of a countable set is countable (or finite). A subset of an uncountable subset NEED NOT be uncountable. Of course, $\mathbb{N}$ is a subset of $\mathbb{R}$!!!! If you wrote this, don't think that you "remembered the theorem wrong..." think that you never internalized the theorem and what it meant.
- Mixing up "infinite" and "uncountable" is a bad error to make.

c) True. This follows from the algebraic limit theorem.

d) True. Zero is an upper bound for the sequence, so this follows from the monotone convergence theorem.

Common mistake: but the limit need not be zero itself. After all, $-12.3$ could also be an upper bound, and so could $-1000$.

e) False. For example, consider the divergent sequence

$$(4.5, 5.5, 4.5, 5.5, 4.5, 5.5, \ldots).$$

Another answer: the sequence could converge to $4.5$. Then only finitely many terms are outside the interval $(4, 5)$.

A common bad answer was: the sequence could converge to $4.5$, so infintely many terms must be inside the interval $(4, 5)$. Just because infintely many terms are inside the interval, does not rule out the possibility that infinitely many terms are also outside the interval!

f) False. (Note the difference between this condition and the Cauchy condition.) For example, $x_n = \log(n)$ is unbounded, but $\log(n+1) - \log(n)$ goes to zero. For another example, try:

$$(0, 1, 1.5, 2, 2\frac{1}{3}, 2\frac{2}{3}, 3, 3\frac{1}{4}, 3\frac{2}{4}, 3\frac{3}{4}, 4, \ldots)$$

so that $(y_n)$ is the sequence

$$(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots).$$

This unbounded sequence goes from $m-1$ to $m$ stepping by $\frac{1}{m}$ at a time, so $y_n$ goes to zero.

If you said this was convergent because it was Cauchy, you missed the purpose of Cauchy's definition.

Another common weird thing: saying that if $y_n$ converges to zero, then the limit of $(x_n)$ agrees with the limit of $(x_{n+1})$. Problem: If $(x_n)$ does not converge, it has no limit, so what is supposed to agree with what? By phrasing things this way, you already assert what you're trying to prove/examine.

6. **(10 pts)** Prove that
$$\lim \frac{7n+2}{2n-5} = \frac{7}{2}.$$

Note that
$$\left|\frac{7n+2}{2n-5} - \frac{7}{2}\right| = \left|\frac{2(7n+2)-(2n-5)(7)}{2(2n-5)}\right| = \left|\frac{39}{4n-10}\right|.$$
Then, so long as $n > 3$, $4n - 10 > 0$, and $|\frac{39}{4n-10}| = \frac{39}{4n-10}$. Now for any $\epsilon > 0$,
$$\frac{39}{4n-10} < \epsilon \iff 4n-10 > \frac{39}{\epsilon} \iff n > \frac{39}{4\epsilon} + \frac{10}{4}.$$
So pick $\epsilon > 0$, and let $N$ be any natural number greater than $\frac{39}{4\epsilon} + \frac{10}{4}$, and also greater than 3. If $n \geq N$, then $n$ is also greater than $\frac{39}{4\epsilon} + \frac{10}{4}$ and 3, meaning that $|\frac{7n+2}{2n-5} - \frac{7}{2}| < \epsilon$, as desired.

Common mistake: starting with
$$\left|\frac{7n+2}{2n-5} - \frac{7}{2}\right| \leq \left|\frac{7n+2}{2n-5}\right| + \left|\frac{7}{2}\right|.$$
This is a big dead end, and no algebra after this will save you. You're trying to show the left hand side is small. But the right hand side is the sum of two large things (7/2 is large, in the scheme of things), so you're never going to use the right hand side to say anything useful.

Common mistake: Forgetting that $2n - 5$ can be negative, so that you have to ensure that $n \geq 3$ to make this term positive and remove the absolute values (also to take the reciprocal and have the inequality flop appropriately). I counted this as a relatively minor error.

Common mistake: Having the implication arrows backward. For some general problem, I don't care if $|x_n - a| < \epsilon \implies n > 46\epsilon$. I'm not trying to show $n$ is big, I'm trying to use $n$ big to prove that $|x_n - a|$ is small. I need the implication $n > 46\epsilon \implies |x_n - a| < \epsilon$. Often this ends up being an if and only if, but writing only one of the arrows is a major logical problem.

Common mistake: Writing the quantifiers (e.g. $\forall$, $\exists$) in the wrong order, or not writing them at all. If $\forall\epsilon > 0$ comes after $\exists N$, then you are saying something totally different.

Common mistake: not having an $N$ anywhere...

Common mistake: Not writing sentences, words, or anything that resembles a proof.

7. **(12 pts)** If $(x_n)$ converges to $101$, and $x_n \neq 0$ for all $n \in \mathbb{N}$, then $\frac{1}{x_n}$ converges to $\frac{1}{101}$. Prove this directly from the definition of convergence (that is, without using the algebraic limit theorem).

Note that
$$\left|\frac{1}{x_n} - \frac{1}{101}\right| = \left|\frac{101 - x_n}{101x_n}\right| = \frac{|101 - x_n|}{101|x_n|}.$$
Since $x_n$ converges to $101$, there is some natural number $N_1$ such that for all $n \geq N_1$ we have $x_n > 100$. Thus for any $n \geq N_1$ we have
$$\frac{|101 - x_n|}{101|x_n|} \leq \frac{|101 - x_n|}{101 \cdot 100} = \frac{|x_n - 101|}{10100}.$$
Pick $\epsilon > 0$. Then there is some natural number $N_2$ such that for all $n \geq N_2$ we have $|x_n - 101| < 10100\epsilon$. Let $N = \max\{N_1, N_2\}$. Then for any $n \geq N$, we see that
$$\left|\frac{1}{x_n} - \frac{1}{101}\right| \leq \frac{|x_n - 101|}{10100} < \frac{10100\epsilon}{10100} = \epsilon,$$
as desired.

Most answers were nowhere close, so the errors were across the map. Many people used some bad algebra to try to solve the problem. Many of the errors of the previous problem were here too. Some new things to watch out for:

Your choice of $N$ had better depend on the sequence $(x_n)$!! The sequence might have $x_n = 10000000$ or $x_n = -\frac{1}{1000000}$ or something, for the first billion terms. So $N$ better be bigger than a billion, in this case. Some $N$ depending only on $\epsilon$ or a bound for $|x_n|$ will not do it.

$\frac{1}{a+b} \neq \frac{1}{a} + \frac{1}{b}$. Moreover, $\frac{1}{a+b} < \frac{1}{a} + \frac{1}{b}$ for $a, b > 0$, while many tried to use the opposite inequality.

Saying that $N$ is bigger than $101\epsilon|x_n|$ doesn't make sense - which $x_n$ do you mean?

If you noticed that some sort of bound was needed to handle the $|x_n|$, that's good. But you need to bound $\frac{1}{|x_n|}$ above, not bound $|x_n|$ above. This involves bounding $|x_n|$ below, which can be done.