Math 316 Practice HW Solution

There was a major mathematical problem in maybe 80 percent of the homeworks: failure to analyze what might happen if the rational number is zero or negative. Is 2^{-1} even? Is 2^0 even? Nope. Just to browbeat you for a minute - I wrote on the assignment: *Be careful, and make sure you have considered every possibility.* This is general advice for students, not often applicable in real life (but for now, classwork is real life) - if a problem explicitly warns you about something, and you haven't found this particular subtlety, then you're probably missing something!

Another common problem: writing $r = \frac{p}{q}$ without saying what p and q are! Maybe being explicit would help the missing cases become clearer.

The biggest problem I found in the language of the writeup was in the start and the conclusion. I'm going to be extremely nitpicky, mostly because it's a short assignment so I don't have too much to go on, and I want the biggest bang for my buck.

In the start:

Many people wrote "Let $r = \frac{p}{q}$ (in lowest terms). Then ..." or "Suppose that $r \in \mathbb{Q}$, $r = \frac{p}{q}$..." but without ever mentioning the hypothesis $2^r = 3$, until they used it a sentence later. If you had previously said "Let $r \in \mathbb{R}$ be the number such that $2^r = 3$ (it exists). Suppose $r \in \mathbb{Q}$..." then this is great, no problem, because r already refers to something.

You'd also be much better off saying "Suppose there exists an $r \in \mathbb{Q}$ such that $2^r = 3$." Then you might not make the error in the conclusion.

In the conclusion:

Many people wrote things like "$2^p \neq 3^q$ so r does not exist" or "2^p is even and 3^q is odd, so r is not a rational number." Several issues - when you reach the contradiction, explicitly say it is a contradiction. I would prefer "but then $2^p \neq 3^q$, a contradiction. Therefore ..." Otherwise it seems like the "nonexistence" of r follows directly from the inequality.

The other problem is the statement "r does not exist" or "r is not a rational number." If you have explicitly said at the beginning "Let $r \in \mathbb{R}$ be the number such that $2^r = 3$." then you can conclude that r is not a rational number. Most people said "Suppose $r = \frac{p}{q}$..." so it is weird to conclude by saying r does not exist or is not a rational number... once you've "popped out" of the supposition, what is r? It's no longer really defined. If you say: "Therefore, *such an* r does not exist." that is way better. You've clarified what doesn't exist: not some nebulous thing r, but an r satisfying the properties mentioned above. Alternatively, "Therefore, no rational number r with $2^r = 3$ can exist."

Here's a solution. The parentheticals are probably not necessary, but again they might have helped you avoid mistakes.

Suppose there exists an $r \in \mathbb{Q}$ with $2^r = 3$. Then $r = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We have

$$2^{\frac{p}{q}} = 3 \implies 2^p = 3^q. \tag{0.1}$$

Now we examine two cases.

If $p > 0$ then 2^p and 3^q are both natural numbers. But 2^p is even (it is 2 times 2^{p-1}, and $2^{p-1} \in \mathbb{N}$), and 3^q is odd (it is a product of odd numbers). Therefore $2^p \neq 3^q$.

If $p \leq 0$ then $2^p \leq 1$, while $3^q \geq 3$. Thus $2^p \neq 3^q$.

In either case, $2^p \neq 3^q$, contradicting (0.1). Thus such a rational number r does not exist.