Math 316 HW6 Comments

Here are some comments from the grader, and some solutions to the problems from HW 6 (series).

1. There was much forgetting of the phrase "there exists $N > 0$ such that for all $n \geq N$..." Most often, students said something like $\lim a_n = \ell$ so $|a_n - \ell| < \epsilon$.

2. When you write
$$\sum a_n$$
the mathematical community interprets this (as in my shorthand in class) as
$$\sum_{n=1}^{\infty} a_n.$$
If you want only a finite sum, you need to indicate the bounds of summands:
$$\sum_{n=1}^{m} a_n = a_1 + a_2 + \ldots + a_m.$$

3. People need to cite the theorems that they use.

4. There are sequences with $\lim a_n = 0$, but (a_n) is not decreasing. There are series with $\lim a_n = 0$ but $\sum a_n$ diverges. Make sure you know examples!

5. Be clear about the difference between a sequence and the individual terms of he sequence. (a_n) is a sequence, and is better written as $(a_n)_{n \in \mathbb{N}}$. While (once n is chosen) a_n is an individual entry of the sequence. Often we omit the parentheses when we write $\lim(a_n)$, that's ok. But perhaps it is better to write $\lim_{n \to \infty} a_n$ to emphasize that the variable n is belonging to a family.

6. Don't mix up a sequence and its series. $\sum \frac{1}{n}$ diverges, but $(\frac{1}{n})$ converges.

7. $\lim(na_n)$ is not equal to $n \lim(a_n)$. The $\lim$ symbol could have been written
$$\lim_{n \to \infty}$$
it has a built-in scope where n is defined. Outside of the limit, n does not make sense!

8. If you see a question asking you to prove $A \iff B$, there must be two parts of the argument:
   - Assume A. Prove B.
   - Assume B. Prove A.

9. In general, more prose would be nice. Explain to me how you got there. Let me know whats going on. What are you proving?

10. Please staple your HW, and write legibly. Your workflow should be easy to follow.

For 2.4.10: here is a proof that $\prod_{i=1}^{\infty}(1+a_i)$ converges if and only if $\sum_{i=1}^{\infty} a_i$ converges, for a sequence $(a_i)_{i\in\mathbb{N}}$ satisfying $a_i \geq 0$ for all i.

Proof. For each $n \in \mathbb{N}$ let $s_n = \sum_{i=1}^{n} a_i$ and $p_n = \prod_{i=1}^{n}(1+a_i)$. Both the sequences (s_n) and (p_n) are increasing thanks to the assumption $a_i \geq 0$, so each sequence converges if and only if it is bounded above, by the MCT. Hence, it is enough to show that (s_n) is bounded if and only if (p_n) is bounded.

Note that, for any $n \in \mathbb{N}$,

$$p_n = (1+a_1)(1+a_2)\cdots(1+a_n) = 1 + a_1 + a_2 + \ldots + a_n + a_1a_2 + \ldots$$

. Every term in this sum is non-negative, so that $p_n \geq 1 + a_1 + a_2 + \ldots + a_n = 1 + s_n$. Since $p_n \geq s_n + 1$ for all $n \in \mathbb{N}$, any upper bound M for the sequence (p_n) is also an upper bound for the sequence (s_n).

As hinted, it is a fact that $1 + a \leq 3^a$ for any $a \geq 0$. Thus

$$p_n = (1+a_1)(1+a_2)\cdots(1+a_n) \leq 3^{a_1}3^{a_2}\cdots 3^{a_n} = 3^{a_1+a_2+\ldots+a_n} = 3^{s_n},$$

for any $n \in \mathbb{N}$. Hence, if M is an upper bound for the sequence (s_n), then 3^M is an upper bound for the sequence (p_n).

Thus (s_n) is bounded above if and only if (p_n) is bounded above. ☐

Proposition 1. Let (a_n) be a sequence with $\lim(na_n) = \ell$ for $\ell \neq 0$. Then $\sum a_n$ diverges.

(Note to students: a common trick, used many times already, and also used on the ratio test in this homework: when you know that some sequence limits to $\ell > 0$, you can't say whether it is above ℓ or below, you can't say how fact it approaches, etcetera etcetera. But you CAN say that it is eventually bigger than ℓ' for some $0 < \ell' < \ell$. One might be more concrete and say that eventually the sequence is larger than $\ell/2$. Similarly, you could say the sequence is eventually smaller than 2ℓ.)

Proof. Suppose that $\ell > 0$ (we will treat the case of $\ell < 0$ later). Pick some ℓ' with $0 < \ell' < \ell$. Then (by letting $\epsilon = \ell - \ell'$) we can choose an $N \in \mathbb{N}$ such that, for all $n \geq N$ we have $na_n > \ell'$. In this case, for any $m > N$ we have

$$\sum_{n=N}^{m} a_n > \sum_{n=N}^{m} \frac{\ell'}{n} = \ell' \sum_{n=N}^{m} \frac{1}{n}.$$

As m goes to infinity, the series $\sum_{n=N}^{m} \frac{1}{n}$ is not bounded above, and thus $\sum_{n=N}^{m} a_n$ is also not bounded above. Since the convergence or divergence of a series depends only on its tail, this means that $\sum_{n=1}^{\infty} a_n$ is also divergent.

If $\ell < 0$, we can make a completely analogous argument. Pick some ℓ' with $\ell < \ell' < 0$. Then eventually $na_n < \ell'$, so that there exists an $N \in \mathbb{N}$ such that, for $m > N$ we have

$$\sum_{n=N}^{m} a_n < \ell' \sum_{n=N}^{m} \frac{1}{n}.$$

As m goes to infinity, the series $\sum_{n=N}^{m} \frac{1}{n}$ is not bounded above, so that $\ell' \sum_{n=N}^{m} \frac{1}{n}$ is not bounded below, and neither is $\sum_{n=N}^{m} a_n$. ☐

I may add additional solutions to this file when I have the time.