Fundamental Group of the Product

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Proposition. *Let $x \in X$, $y \in Y$, and*

$$\begin{tikzcd} X \times Y \arrow[r, "q"] \arrow[d, "p"'] & Y \\ X & \end{tikzcd}$$

be the natural maps. Then the map

$$\begin{aligned} \Phi : \pi_1(X \times Y, (x, y)) &\to \pi_1(X, x) \times \pi_1(Y, y) \\ [\gamma] &\mapsto (p_*[\gamma], q_*[\gamma]) \end{aligned}$$

is an isomorphism.

Proof. That Φ is a group homomorphism is automatic from definition of the product group and the fact that p_* and q_* are group homomorphisms.

First we show that Φ is surjective. Given $([\alpha], [\beta]) \in \pi_1(X, x) \times \pi_1(Y, y)$, define $\gamma : I \to X \times Y$ by $\gamma(s) = (\alpha(s), \beta(s))$. Then $\gamma(0) = \gamma(1) = (x, y)$, so $[\gamma] \in \pi_1(X \times Y, (x, y))$, and we have $p \circ \gamma = \alpha$ and $q \circ \gamma = \beta$, so $\Phi([\gamma]) = (p_*[\gamma], q_*[\gamma]) = ([\alpha], [\beta])$.

Second we show that Φ is injective. Suppose that we have $[\gamma_0], [\gamma_1] \in \pi_1(X \times Y, (x, y))$ with $\Phi([\gamma_0]) = \Phi([\gamma_1])$. Then $p_*[\gamma_0] = p_*[\gamma_1]$, so $p \circ \gamma_0 \simeq p \circ \gamma_1$, so let $F : I \times I \to X$ a homotopy rel. endpoints from $p \circ \gamma_0$ to $p \circ \gamma_1$. Similarly, let $G : I \times I \to Y$ a homotopy rel. endpoints from $q \circ \gamma_0$ to $q \circ \gamma_1$. Then the map $H : I \times I \to X \times Y$ defined by $H(s, t) = (F(s, t), G(s, t))$ is a homotopy rel. endpoints from γ_0 to γ_1, so $[\gamma_0] = [\gamma_1]$. $\square$