Homework 9

Due Monday, March 19, 2011

I should have mentioned in lecture that if k is a field then for any map $f : X \to Y$, the induced map $f_* : H_i(X; k) \to H_i(Y; k)$ is k-linear; that is, it is a homomorphism of vector spaces, not merely of groups.* The same is true of the connecting homomorphism ∂ in the long exact sequence of the pair, etc.

1. Let X be an n-dimensional cell complex, and let m_i be the number of i-cells. Consider the cellular chain complex with rational coefficients:

   $$0 \to \mathbb{Q}^{m_n} \xrightarrow{d_n} \cdots \xrightarrow{d_{i+1}} \mathbb{Q}^{m_i} \xrightarrow{d_i} \cdots \xrightarrow{d_2} \mathbb{Q}^{m_1} \xrightarrow{d_1} \mathbb{Q}^{m_0} \to 0.$$

   Let r_i be the rank of d_i, that is, $r_i = \dim(\operatorname{im} d_i)$.

   (a) What is the dimension of $\ker d_i$?

   (b) What is the dimension of $H_i(X; \mathbb{Q})$?

   (c) Show that $m_0 - m_1 + m_2 - \cdots \pm m_n = \chi(X)$.

2. Using the previous problem, find the Euler characteristic of an orientable surface of genus g (that is, a g-holed torus), and of the non-orientable surface $\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ (k times).

3. Suppose that a polyhedron with v vertices, e edges, and f faces is homeomorphic to S^2. Argue that $v - e + f = 2$. Make a table of v, e, and f for the five Platonic solids.

4. What is one question you have about last week's lectures?

*More generally, if R is a ring then $H_i(X; R)$ and $H_i(Y; R)$ are R-modules and f_* is an R-module homomorphism.