Homework 7

Due Monday, March 5, 2011

1. A pair of homomorphisms of abelian groups

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

   is called exact (at B) if $\operatorname{im}\varphi = \ker\psi$. A sequence of homomorphisms

$$A_n \to A_{n-1} \to A_{n-2} \to \cdots \to A_0$$

   is called exact if it is exact at each A_i (except A_n and A_0, where that doesn't make sense).

   (a) Show that $0 \to A \xrightarrow{\varphi} B$ is exact iff φ is injective.

   (b) Show that $B \xrightarrow{\psi} C \to 0$ is exact iff ψ is surjective.

   (c) In what sense is an exact sequence $0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ equivalent to the statement $C = B/A$?

   (d) Suppose that $A \xrightarrow{\varphi} B \xrightarrow{\chi} C \xrightarrow{\psi} D$ is exact. Show that χ is an isomorphism iff $\varphi = 0$ and $\psi = 0$.

2. The direct sum $A \oplus B$ of two abelian groups is the same as the direct product $A \times B$.* If A and B are subgroups of an abelian group C, they generate a subgroup

$$A + B = \{a + b \mid a \in A, b \in B\} \subset C.$$

   If $A + B = C$ and $A \cap B = 0$, exhibit an isomorphism $A \oplus B \to C$.

3. What is one question you have about last week's lectures?

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*The direct sum of infinitely many abelian groups is different from the direct product: an element of $\prod_{i=1}^{\infty} A_i$ is a sequence $(a_1, a_2, a_3, \dots)$ with $a_i \in A_i$, while an element of $\bigoplus_{i=1}^{\infty} A_i$ is such a sequence where all but finitely many a_i's are zero.