Homework 3

Due Monday, January 30, 2012

1. Let $x \in X$, let $\gamma : I \to X$ be a path with $\gamma(0) = \gamma(1) = x$, and similarly γ_0, γ', etc. Let $\simeq$ denote homotopy rel. endpoints and $\cdot$ denote concatenation of paths.

   (a) Prove one of the following:

      i. Multiplication in $\pi_1(X, x)$ is well-defined: if $\gamma_0 \simeq \gamma_1$ and $\gamma_0' \simeq \gamma_1'$ then $\gamma_0 \cdot \gamma_0' \simeq \gamma_1 \cdot \gamma_1'$.

      ii. The inverse is well-defined: if $\gamma_0 \simeq \gamma_1$ then $\gamma_0^{-1} \simeq \gamma_1^{-1}$. Here γ_i^{-1} denotes the reverse path, not the preimage.

   (b) Prove one of the following:

      i. Associativity axiom: $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \simeq \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$.

      ii. Identity axiom: $\gamma \cdot e \simeq \gamma$. At the end of your proof write, "Similarly, $e \cdot \gamma \simeq \gamma$."

      iii. Inverse axiom: $\gamma \cdot \gamma^{-1} \simeq e$.

   To prove that two paths are homotopic rel. endpoints, you must write down the homotopy explicitly, not just draw a picture.

2. Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$. Hint: A disconnected space is not homeomorphic to a connected one.

3. What is one question you have about last week's lectures?