Homework 1

Due Monday, January 16, 2012

You may work with other students on these problems, but you must write them up yourself, in your own words. Use pencil, and double space (skip lines) to leave room for my comments. If you type, use TEX, not Microsoft Word. Do not use the shorthand ∀ and ∃; you have time to write "for every" and "there is."

1. (a) Let X and Y be sets and $f : X \to Y$ a surjection. Show that for any other set Z, a map $g : X \to Z$ descends to Y—that is, there is a map $h : Y \to Z$ with $g = h \circ f$—if and only if it is constant on the fibres of f.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "h"'] & Z \end{tikzcd}$$

   (b) Now let X and Y be spaces and $f : X \to Y$ a quotient map. Show that for any other space Z, a continuous map $g : X \to Z$ descends to Y if and only if it is constant on the fibres of f.

2. (a) For each of the following, indicate whether you are familiar with the fact, and familiar with the proof:
      i. If X is compact and $F \subset X$ is closed then F is compact.
      ii. If X is compact and $f : X \to Y$ is continuous then the image $f(X) \subset Y$ is compact.
      iii. If Y is Hausdorff and $K \subset Y$ is compact then K is closed.

   (b) Suppose that X is compact, Y is Hausdorff, and $f : X \to Y$ is a continuous surjection. Show that f is a quotient map. You may quote any of the facts from part (a).

3. I claim that $(S^1 \times I)/(S^1 \times \{0\})$ is homeomorphic to D^2.
   (a) Draw a picture.
   (b) Argue that it is enough to write down a continuous surjection $S^1 \times I \to D^2$ which is constant on $S^1 \times \{0\}$ and injective otherwise.
   (c) Write down such a map.