Homework 6

Due Friday, March 4, 2011

Recall that $\mathbf{Top}_*$ is the category of pointed spaces. Objects are pairs (X, p) where X is a topological space and $p \in X$. Arrows from (X, p) to (Y, q) are continuous functions $f : X \to Y$ with $f(p) = q$.

If (X, p) is a pointed space, define the *reduced suspension* ΣX to be the quotient space

$$\Sigma X = \frac{X \times I}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{p\} \times I)}.$$

It is a pointed space; the basepoint is the equivalence class of $(p, 0)$.

Define the *based loop space* ΩX to be the set of continuous maps $\gamma : I \to X$ with $\gamma(0) = \gamma(1) = p$, endowed with the compact-open topology, which we will not define.

In these exercises we will show that Σ and Ω are functors from $\mathbf{Top}_*$ to $\mathbf{Top}_*$ and that they are adjoint to one another:

$$\mathrm{Hom}_{\mathbf{Top}_*}(\Sigma X, Y) = \mathrm{Hom}_{\mathbf{Top}_*}(X, \Omega Y).$$

1. Draw pictures of ΣS^0 and ΣS^1.
2. What is ΣS^n? (Do not work hard on this problem.)
3. Given a pointed map $f : (X, p) \to (Y, q)$, what should be the induced map $\Sigma f : \Sigma X \to \Sigma Y$? (In one sense there are many possible answers to this question, but really there is only one right answer.) Show (or just point out, if it is obvious) that the Σf you define is continuous and respects basepoints.
4. What should be the basepoint of ΩX?
5. What should be the induced map $\Omega f : \Omega X \to \Omega Y$? Since we are being vague about the topology on the loop space, you need not show that Ωf is continuous, but it should respect basepoints.
6. Given a pointed map $f : \Sigma X \to Y$, what should be the corresponding map $g : X \to \Omega Y$?
7. Given a pointed map $g : X \to \Omega Y$, what should be the corresponding map $f : \Sigma X \to Y$?
8. Σ and Ω induce functors from the homotopy category $\mathbf{Toph}_*$ to itself, and these too are adjoint. Think about what else you would have to do to check this.