Homework 3

Due Friday, February 4, 2011

1. Let (X, p) be a pointed space, $\gamma : I \to X$ a path with $\gamma(0) = \gamma(1) = p$, and similarly γ_0, γ', etc. Let $\simeq$ denote homotopy rel. endpoints and $\cdot$ denote concatenation of paths. Prove any two of the following:

   (a) Multiplication in $\pi_1(X, p)$ is well-defined: if $\gamma_0 \simeq \gamma_1$ and $\gamma'_0 \simeq \gamma'_1$ then $\gamma_0 \cdot \gamma'_0 \simeq \gamma_1 \cdot \gamma'_1$.

   (b) The inverse is well-defined: if $\gamma_0 \simeq \gamma_1$ then $\gamma_0^{-1} \simeq \gamma_1^{-1}$.

   (c) Associativity axiom: $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \simeq \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$.

   (d) Identity axiom: $\gamma \cdot e \simeq \gamma$. At the end of your proof write, "Similarly, $e \cdot \gamma \simeq \gamma$."

   (e) Inverse axiom: $\gamma \cdot \gamma^{-1} \simeq e$.

   To prove that two paths are homotopic rel. endpoints, you must write down the homotopy explicitly, not just draw a picture.

2. Let $p, q \in X$ and let α be a path from p to q. Let α^{-1} denote the reverse path, that is, $\alpha^{-1}(t) = \alpha(1 - t)$. Show that the map

$$\begin{aligned} \pi_1(X, q) &\to \pi_1(X, p) \\ \gamma &\mapsto \alpha \cdot \gamma \cdot \alpha^{-1} \end{aligned}$$

   is a well-defined isomorphism. (Feel free to say things like $\alpha \cdot \alpha^{-1} \simeq e_p$, where e_p is the constant loop at p, without proof; even though $\alpha(0) \neq \alpha(1)$, the proof is the same as it was for loops. But be careful not to concatenate two paths unless the second one begins where the first one ends.)

3. Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$. Hint: A disconnected space is not homeomorphic to a connected one.