Homework 2

Due Friday, January 28, 2011

1. Let X and Y be topological spaces. Show that homotopy is an equivalence relation on $\mathrm{Hom}(X, Y)$.

2. Let X, Y, and Z be topological spaces, and $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, show that $g_0 \circ f_0 \simeq g_1 \circ f_1$.

3. Let X be a topological space. Show that the following are equivalent:

   (a) X is contractible.

   (b) For every space Y, every map $f : X \to Y$ is nullhomotopic.

   (c) For every space Z, every map $g : Z \to X$ is nullhomotopic.

   Hint: Use the proposition proved in lecture on Friday.

4. Let X be a topological space. Define $x \sim x'$ if there is a path from x to x'. Convince yourself that $\sim$ is an equivalence relation (but do not turn this in). The equivalence classes of $\sim$ are called the *path components* of X. We define $\pi_0(X) = X/{\sim}$.

   (a) Show that a map $f : X \to Y$ induces a map $\pi_0(X) \to \pi_0(Y)$.

   (b) Show that homotopic maps $X \to Y$ induce the same map on π_0.

   (c) Show that a homotopy equivalence induces a bijection on π_0.

5. Let X be a topological space and $f : S^1 \to X$. Show that the following are equivalent:

   (a) f is nullhomotopic.

   (b) f extends to a map $D^2 \to X$; that is, there is a map $g : D^2 \to X$ such that the restriction $g|_{S^1} = f$.