Homework 1

Due Friday, January 21, 2011

You may work with other students on these problems, but you must write them up yourself, in your own words. Use pencil and double psace (skip lines). If you type, use TeX, not Microsoft Word. Do not use the shorthand $\forall$ and $\exists$; you have time to write "for every" and "there is."

This homework looks longer than it is. Your solutions may well be shorter than the assignment.

1. Let X be a set. Recall that:
   - An *equivalence relation* on X is a relation $\sim$ that is reflexive ($x \sim x$), symmetric (if $x \sim y$ then $y \sim x$), and transitive (if $x \sim y$ and $y \sim z$ then $x \sim z$).
   - A *partition* of X is a collection of disjoint subsets of X whose union is all of X.
   - A *surjection* from X to another set is an "onto" map.

   (a) Show that an equivalence relation on X determines a partition of X, and vice versa.

   (b) Show that an equivalence relation on X determines a surjection onto another set, and vice versa.

2. Let X be a topological space.

   (a) Let A be a set and $i : A \to X$ an injection. Define the subspace topology on A.

   (b) (Universal property of the subspace topology.) If we give A the subspace topology then i is a continuous map whose image is contained $i(A)$. Show that any other map with this property factors through i: that is, if $f : Z \to X$ is a continuous map whose image is contained in $i(A)$ then there is a unique continuous map $g : Z \to A$ with $f = i \circ g$.

$$\begin{tikzcd} Z \arrow[d, dotted, "g"'] \arrow[dr, "f"] & \\ A \arrow[r, "i"'] & X \end{tikzcd}$$

(c) Let Y be a set and $q : X \to Y$ be a surjection. Define the quotient topology on Y.

(d) (Universal property of the quotient topology.) Let $\sim$ be the equivalence relation on X determined by q as problem 1(b). If we give Y the quotient topology then q is a continuous map such that $q(x) = q(x')$ whenever $x \sim x'$. Show that any other map with this property factors through q: that is, if $f : X \to Z$ is a continuous map such that $f(x) = f(x')$ whenever $x \sim x'$ then there is a unique continuous map $g : Y \to Z$ with $f = g \circ q$.

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "f"'] & Y \arrow[d, dotted, "g"] \\ & Z \end{tikzcd}$$

3. Let X be a topological space, $\sim$ an equivalence relation on X, and Y another topological space. Show that the following are equivalent:

(a) The quotient space $X/\sim$ is homeomorphic to Y.

(b) There is a surjection $q : X \to Y$ such that $q(x) = q(x')$ if and only $x \sim x'$, and $U \subset Y$ is open if and only if $q^{-1}(U) \subset X$ is open.

4. I claim that $(S^1 \times I)/(S^1 \times \{0\})$ is homeomorphic to D^2.

(a) Draw a picture.

(b) Write down the appropriate map $S^1 \times I \to D^2$. You need not show that it is a quotient map, but it should be obviously continuous.