Enhanced Coursework

Due May 14, 2011

1. (a) Draw pictures of $S^2 \vee S^1$ and its universal cover.
   (b) Let X be S^2 union a line segment connecting the north and south poles. Draw pictures of X and its universal cover.
2. Let G be a finite group. Show that any homomorphism $\varphi : G \to \mathbb{Z}$ is zero.
3. The Borsuk–Ulam theorem states that for any map $f : S^2 \to \mathbb{R}^2$ there is an $x \in S^2$ with $f(-x) = f(x)$. The proof reduces to the following lemma: there is no odd map $g : S^2 \to S^1$, i.e. no map with $g(-x) = -g(x)$ for all $x \in S^2$. In lecture we proved this lemma by showing that an odd map $S^1 \to S^1$ induces a nonzero map on π_1, hence is not nullhomotopic, and considered g restricted to the equator to get a contradiction. Give a second proof of the lemma, as follows.
   (a) Let $q : S^2 \to \mathbb{RP}^2$ be the universal cover and $p : S^1 \to S^1$ the double cover $p(z) = z^2$. Observe that an odd map $g : S^2 \to S^1$ descends to a map h as in this diagram:
   $$\begin{tikzcd} S^2 \arrow[r, "g"] \arrow[d, "q"'] & S^1 \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[r, dotted, "h"'] & S^1. \end{tikzcd}$$
   (b) Use the lifting criterion to produce a map $\tilde{h} : \mathbb{RP}^2 \to S^1$ lifting h.
   $$\begin{tikzcd} & S^1 \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[ur, "\tilde{h}"] \arrow[r, dotted, "h"'] & S^1 \end{tikzcd}$$
   (c) Use the unique lifting property to show that $\tilde{h} \circ q = g$, which contradicts the assumption that g was odd.
4. Recall that **Toph**$_*$ is the category whose objects are pointed spaces and whose arrows are homotopy classes of maps (homotopic rel. basepoint), and **Grp** is the category whose objects are groups and arrows are group homomorphisms.

   A map $f : (X, x_0) \to (S^1, 1)$ induces a map
   $$f_* : \pi_1(X, x_0) \to \pi_1(S^1, 1) = \mathbb{Z}.$$

Show that the map

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Toph}_*}((X,x_0),(S^1,1)) &\to \mathrm{Hom}_{\mathbf{Grp}}(\pi_1(X,x_0),\mathbb{Z}) \\ f &\mapsto f_* \end{aligned} \tag{$*$}$$

is injective, as follows.

(a) Let G be any group, $e \in G$ the identity, and $\varphi : G \times G \to G$ a homomorphism with $\varphi(g,e) = g = \varphi(e,g)$ for all $g \in G$. [At this point I meant to write "Conclude that $\varphi(g,h) = gh$ for all $g,h \in G$." Many of you figured this out.] (Optional: Conclude that G is abelian.)

(b) Let $m : S^1 \times S^1 \to S^1$ be the multiplication map $m(z,w) = zw$. Apply part (a) to $m_* : \pi_1(S^1) \times \pi_1(S^1) \to \pi_1(S^1)$ to conclude that m_* is the multiplication in $\pi_1(S^1)$. (We are dealing with several groups here: S^1 with complex multiplication, $\pi_1(S^1,1)$ with concatenation of paths, and $\mathbb{Z}$ with addition. Do not get confused about which group you are in.)

(c) Define a group structure on the left-hand side of $(*)$. (Use m to multiply elements of S^1 or we'll all go nuts.)

(d) Define a group structure on the right-hand side of $(*)$.

(e) Show that the map in $(*)$ is a group homomorphism.

(f) Let (X,x_0) be any pointed space and $f : (X,x_0) \to (S^1,1)$ a map such that $f_* : \pi_1(X,x_0) \to \mathbb{Z}$ is the zero map. Show that f is nullhomotopic. (Hint: Consider the universal cover $p : \mathbb{R} \to S^1$.) Conclude that the map in $(*)$ is injective.

5. Optional:* Suppose that X is good in the sense that it has a universal cover and admits partitions of unity. Show that the map in $(*)$ above is also surjective, as follows. Let $G = \pi_1(X,x_0)$ and $p : \tilde{X} \to X$ be the universal cover. Recall that $\tilde{X}$ looks locally like $X \times G$; that is, we can choose an open cover $\{U_\alpha\}$ of X and maps $\psi_\alpha : p^{-1}(U_\alpha) \to G$, each of which gives a bijection with between the fibres of p and G. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the cover $\{U_\alpha\}$.

   Now given a homomorphism $\varphi : G \to \mathbb{Z}$, define a map $F : \tilde{X} \to \mathbb{R}$ by

$$F(\tilde{x}) = \sum_\alpha \rho_\alpha(p(\tilde{x})) \cdot \varphi(\psi_\alpha(\tilde{x}))$$

   Observe that this is well-defined. Show that it descends to a map $f : X \to S^1$ with $f_* = \varphi$.

*Really. You have better things to do with your time than work on this problem.