Homework 3

Due March 19, 2015

1. Some multilinear algebra

Do this cleanly – don't make a big mess of it, and don't belabor easy stuff. Choose a basis when it makes the argument cleaner; don't when it doesn't.

Let V be an n-dimensional $\mathbb{R}$-vector space. Recall that

$$\Lambda V = \bigoplus_{k=0}^{n} \Lambda^k V$$

is the $\mathbb{R}$-algebra generated by V subject to the relation

$$v \wedge w = -w \wedge v.$$

Thus we are regarding $\Lambda^k V$ as a quotient of $V \otimes \cdots \otimes V$, not as a subspace. Note that a general element of $\Lambda^k V$ is a *sum* of terms of the form $v_1 \wedge \cdots \wedge v_k$.

(a) Consider the pairing

$$\Lambda^k(V^*) \otimes \Lambda^k V \to \mathbb{R}$$

determined by

$$\langle \varphi_1 \wedge \cdots \wedge \varphi_k,\ v_1 \wedge \cdots \wedge v_k \rangle = \det \begin{pmatrix} \varphi_1(v_1) & \cdots & \varphi_1(v_k) \\ \vdots & \ddots & \vdots \\ \varphi_k(v_k) & \cdots & \varphi_k(v_k) \end{pmatrix}.$$

Say briefly why this is well-defined. Show that it is non-degenerate; that is, for every $\omega \in \Lambda^k(V^*)$ there is an $X \in \Lambda^k V$ such that $\langle \omega, X \rangle \neq 0$. Thus the map

$$\begin{array}{ccc} \Lambda^k(V^*) & \to & (\Lambda^k V)^* \\ \omega & \mapsto & \langle \omega, - \rangle \end{array}$$

is an isomorphism.

(b) For $l \leq k$, define contraction

$$\lrcorner \colon \Lambda^l V \otimes \Lambda^k V^* \to \Lambda^{k-l} V^*$$

as the adjoint of wedging

$$\wedge \colon \Lambda^l V \otimes \Lambda^{k-l} V \to \Lambda^k V.$$

That is, for $X \in \Lambda^l V$ and $Y \in \Lambda^{l-k} V$ and $\omega \in \Lambda^k V^*$ we have

$$\langle X \lrcorner \, \omega, Y \rangle = \langle \omega, X \wedge Y \rangle.$$

Given $v, w \in V$ and $\varphi_1, \ldots, \varphi_k \in V^*$, check that

$$v \lrcorner (\varphi_1 \wedge \cdots \wedge \varphi_k) \qquad \text{and} \qquad (v \wedge w) \lrcorner (\varphi_1 \wedge \cdots \wedge \varphi_k)$$

agree with what you think they should be.

(c) Fix a non-zero $vol \in \Lambda^n V^*$. Show that map

$$\begin{aligned} \Lambda^k V &\to \Lambda^{n-k} V^* \\ X &\mapsto X \lrcorner \, vol \end{aligned}$$

is an isomorphism.

(d) Fix an inner product g on V. This determines an isomorphism

$$\begin{aligned} V &\xrightarrow{\tilde{g}} V^* \\ v &\mapsto g(v, -) \end{aligned}$$

and thus an isomorphism

$$\begin{aligned} \Lambda^k V &\xrightarrow{\tilde{g}} \Lambda^k V^* \\ v_1 \wedge \cdots \wedge v_k &\mapsto g(v_1, -) \wedge \cdots \wedge g(v_k, -) \end{aligned}$$

for each k. Combining this isomorphism with the pairing of part (a) we get pairings on $\Lambda^k V$ and $\Lambda^k V^*$, which we will also call g. Check that these are inner products (symmetric and positive definite).

(e) For $X \in \Lambda^k V$, consider both

$$g(X, X) \cdot vol \qquad \text{and} \qquad (X \lrcorner \, vol) \wedge \tilde{g}(X)$$

in $\Lambda^n V^*$. How do they compare? Given g, is there a best choice for vol?

(f) Optional: If you have anything else to say about these operations on ΛV and ΛV^*, say it.

2. Connections as splittings

(a) Let $\pi \colon E \to M$ be a vector bundle. There is a canonical section of the vector bundle $\pi^* E$ on E. What is it?

(b) Recall that a connection on E determines a splitting of the natural exact sequence

$$0 \to \pi^* E \to TE \xrightarrow{D\pi} \pi^* TM \to 0$$

of vector bundles on E. Take $E = TM$, so this becomes

$$0 \to \pi^* TM \to TTM \xrightarrow{D\pi} \pi^* TM \to 0.$$

With a splitting of this sequence we can take the canonical section of the right-hand $\pi^* TM$ and get a section of the middle term TTM, i.e. a vector field on TM. Show that this is the geodesic spray: that is, if $\gamma : (0,1) \to M$ is a curve and $\tilde{\gamma} : (0,1) \to TM$ is the lift discussed in class, then γ is a geodesic if and only if $\tilde{\gamma}$ is an integral curve of this vector field on TM.

(c) Optional: Explore what it means for a connection on TM to be symmetric, in terms of the splitting of

$$0 \to \pi^* TM \to TTM \xrightarrow{D\pi} \pi^* TM \to 0.$$