Solutions to Homework 3

Math 487

October 7, 2013

VI.4.10 Let $P = \{2, 3, 5, 7, 11, \ldots\}$ be the set of prime numbers. Let $S = \{0, 1, +, \times\}$, and let $\mathfrak{N}$ be the standard S-structure on $\mathbb{N}$. Given an S-structure $\mathfrak{A} \models \mathrm{Th}(\mathfrak{N})$ and an element $a \in A$, we define

$$\mathrm{div}(a, \mathfrak{A}) := \{p \in P \mid \text{there is a } b \in A \text{ such that } a = \overbrace{b +^{\mathfrak{A}} \cdots +^{\mathfrak{A}} b}^{p \text{ times}}\},$$

i.e. the set of primes dividing a according to $\mathfrak{A}$

First we argue that for any subset $Q \subset P$ there is a countable S-structure $\mathfrak{A} \models \mathrm{Th}(\mathfrak{N})$ and an element $a \in A$ with $\mathrm{div}(a, \mathfrak{A}) = Q$. For each $p \in P$, consider the formula

$$\varphi_p := \exists v_1\ v_0 = \overbrace{v_1 + \cdots + v_1}^{p \text{ times}},$$

which says that v_0 is divisible by p. Given a subset $Q \subset P$, let

$$\Phi_Q = \mathrm{Th}(\mathfrak{N}) \cup \{\varphi_p \mid p \in Q\} \cup \{\neg\varphi_p \mid p \in P \setminus Q\}.$$

Every finite subset $\Psi \subset \Phi_Q$ is satisfiable, as follows. We have

$$\Psi \subset \mathrm{Th}(\mathfrak{N}) \cup \{\varphi_{p_1}, \ldots, \varphi_{p_m}\} \cup \{\neg\varphi_{p_{m+1}}, \ldots, \neg\varphi_{p_{m+n}}\}$$

for some distinct primes $p_1, \ldots, p_{m+n}$. The interpretation consisting of the structure $\mathfrak{N}$ and the assignment $\beta(v_i) = p_1 \cdots p_m \in \mathbb{N}$ satisfies Ψ, as claimed. Thus every finite subset of Φ_Q is consistent, so Φ_Q is consistent, so there is a countable interpretation $(\mathfrak{A}, \beta') \models \Phi_Q$. Now $a := \beta'(v_0) \in A$ has the required property.[1]

[1] Note that Q may be empty; in that case our proof says to take $\beta(v_i) = 1$ for all finite subsets $\Psi \subset \Phi_Q$, which is no problem. Thus there is a non-standard model of arithmetic containing an infinite element a which is not divisible by any finite element!

Next we argue that there are uncountably many non-isomorphic countable of $\mathrm{Th}(\mathfrak{N})$. Suppose that $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \ldots$ are countable models of $\mathrm{Th}(\mathfrak{N})$ with $\mathfrak{A}_i \not\cong \mathfrak{A}_j$ for $i \neq j$; we will produce a countable model $\mathfrak{A} \vDash \mathrm{Th}(\mathfrak{N})$ which is not isomorphic to any $\mathfrak{A}_i$. Each A_i is countable, and a countable union of countable sets is countable,[2] so only countably many subsets of P are of the form $\mathrm{div}(a, \mathfrak{A}_i)$ for some i and some $a \in A_i$. But P is infinite, so the set of subsets of P is uncountable as we showed in exercise I.1.5. Hence there is a $Q \subset P$ which is not of the form $\mathrm{div}(a, \mathfrak{A}_i)$ for some i and some $a \in A_i$. But we saw above that there is a countable model $\mathfrak{A} \vDash \mathrm{Th}(\mathfrak{N})$ and an element $a \in A$ with $\mathrm{div}(a, \mathfrak{A}) = Q$; hence $\mathfrak{A}$ is not isomorphic to any $\mathfrak{A}_i$, as required.

Problem on ordinals:

(a) Rewrite the formula $x = y \cup \{y\}$ using only $\in$.

$$\forall z\, (z \in x \leftrightarrow (z \in y \vee z \equiv y))$$

(b) Write a formula using only $\in$ that means "x is an ordinal."

$$\forall y\, (y \in x \to \forall z\, (z \in y \to z \in x))$$

(c) Write a formula using only $\in$ that means "x is a finite ordinal." There are many equally good answers, but here is one:

$$\Big(\overbrace{\forall y\, (y \in x \to \forall z\, (z \in y \to z \in x))}^{x \text{ is an ordinal}} \wedge$$
$$\forall y \Big(\big(\overbrace{\forall z\, (z \in y \to z \in x)}^{y \subset x} \wedge \overbrace{\forall z\, (z \in y \to \forall w\, (w \in z \to w \in y))}^{y \text{ is an ordinal}} \big) \to$$
$$\big(\overbrace{\neg \exists z\, z \in y}^{y = \emptyset} \vee \exists z\, \overbrace{\forall w\, (w \in y \leftrightarrow (w \in z \vee w \equiv z))}^{y = z \cup \{z\}} \big) \Big) \Big)$$

This could be optimized in various ways, at the cost of being even more opaque; for example the two occurences of $\forall y$ could be replaced with one $\forall y$ before the first parenthesis, or in the second line we could combine the two $\forall z\, (z \in y \to \cdots)$ pieces into one:

$$\forall z\, (z \in y \to (z \in x \wedge \forall w\, (w \in z \to w \in y)),$$

which says y is an ordinal contained in x.

[2] Here we have used the axiom of choice, but with more care (i.e. more fuss) we could have avoided doing so.