Solutions to Homework 1

Math 487

September 6, 2013

1.3 Following the hint, we construct a nested sequence of intervals recursively. Let $I_0 = [a, b]$. Now suppose that $I_n = [a_n, b_n]$ has been constructed. If $\alpha(n) \notin I_n$, let $I_{n+1} = I_n$. If $\alpha(n) = a_n$, let $I_{n+1} = [\frac{\alpha(n)+b_n}{2}, b_n]$. Otherwise, let $I_{n+1} = [a_n, \frac{a_n+\alpha(n)}{2}]$. Then $I_{n+1} \subset I_n$, so by the Nested Interval Theorem (which boils down to the least upper bound property of $\mathbf{R}$) there is a $c \in \bigcap I_n$. For each n we have $\alpha(n) \notin I_{n+1}$, so $\alpha(n) \neq c$. Thus $\alpha : \mathbf{N} \to \mathbf{R}$ is not surjective. Since α was arbitrary, we conclude that $\mathbf{R}$ is uncountable.

1.4(a) Since each M_n is at most countable, choose a surjection $\alpha_n : \mathbf{N} \to M_n$ for each n. (Note that we use the axiom of choice to make infinitely many choices at once here.) The sequence

$$\begin{aligned}
&\alpha_0(0)\\
&\alpha_0(1), \alpha_1(0),\\
&\alpha_0(2), \alpha_1(1), \alpha_2(0)\\
&\alpha_0(3), \alpha_1(2), \alpha_2(1), \alpha_3(2)\\
&\ldots\\
&\alpha_0(n), \alpha_1(n-1), \ldots, \alpha_n(0)\\
&\ldots.
\end{aligned}$$

defines a surjection $\mathbf{N} \to \bigcup M_n$, so the latter is at most countable.

(b) Order the symbols of $\mathcal{A}$, and let M_n be the set of all strings in $\mathcal{A}^*$ of length at most n which contain at most the first n symbols of $\mathcal{A}$. Then $\mathcal{A}^* = \bigcup M_n$ is a countable union of at most countable sets (finite sets in fact), hence is at most countable. As in the proof of Lemma 1.2, $\mathcal{A}^*$ cannot be finite since contains strings of arbitrary length, so it is countable.

1.5 Given any map $\alpha : M \to \mathcal{P}(M)$, let $R = \{a \in M \mid a \notin \alpha(a)\}$. Then there is no $a \in M$ such that $\alpha(a) = R$, for if there were, we would have $a \in R$ iff $a \notin \alpha(a) = R$, a contradiction. Hence α is not surjective.

4.6(a) First, say that a string has property P if it is of the form xt, where x is a variable, t is a term, and $x \in \mathrm{var}(t)$. We want to show that if xt is derivable in $\mathfrak{C}_v$ then xt has property P. As is discussed at the end of p. 19, it is enough to show that everything derivable in $\mathfrak{C}_v$ by means of a premise-free rule has property P, and that the property P is preserved under the application of the remaining rules. There is one premise-free rule in $\mathfrak{C}_v$, namely

$$\frac{}{x\ x},$$

and indeed $x \in \mathrm{var}(x) = \{x\}$, i.e. xx has property P. There is one other rule, namely,

$$\frac{y\ t_i}{y\ ft_1 \cdots t_n},$$

and if $y \in \mathrm{var}(t_i)$ for some $i \in \{1, \dots, n\}$, i.e. if the premise has property P, then $y \in \mathrm{var}(ft_1 \dots t_n) = \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_n)$, i.e. the conclusion has property P.

Next, fix a variable x say that a term t has property Q if either $x \notin \mathrm{var}(t)$ or the string xt is derivable in $\mathfrak{C}_v$. We want to show that if $x \in \mathrm{var}(t)$ then xt is derivable in $\mathfrak{C}_v$, or equivalently that every term t has property Q. We will prove the latter by induction on the calculus of terms:

$$\frac{}{x} \tag{T1}$$

$$\frac{}{c} \tag{T2}$$

$$\frac{t_1, \dots, t_n}{ft_1 \cdots t_n}. \tag{T3}$$

For the premise-free rule (T1), the conclusion x has property Q since xx is derivable by the first rule of $\mathfrak{C}_v$. For the premise-free rule (T2), the conclusion c has property Q since $x \notin \mathrm{var}(c) = \emptyset$. For the remaining rule (T3), suppose that $t_1, \dots, t_n$ have property Q. If $x \notin \mathrm{var}(t_i)$ for all $i \in \{1, \dots, n\}$ then $x \notin \mathrm{var}(ft_1 \cdots t_n) = \mathrm{var}(t_1) \cup \cdots \cup \mathrm{var}(t_n)$, so $ft_1 \cdots t_n$ has property Q. Otherwise we have $x \in \mathrm{var}(t_i)$ for some $i \in \{1, \dots, n\}$, so $ft_1 \cdots t_n$ has property Q by the second rule of $\mathfrak{C}_v$.

(b)

$$\frac{}{\varphi\ \varphi} \qquad \frac{\eta\ \varphi}{\eta\ \neg\varphi}$$

$$\frac{\eta\ \varphi}{\eta\ (\varphi * \psi)} \qquad \frac{\eta\ \psi}{\eta\ (\varphi * \psi)} \qquad \text{for } * = \vee, \wedge, \rightarrow, \leftrightarrow$$

$$\frac{\eta\ \varphi}{\eta\ \forall x\varphi} \qquad \frac{\eta\ \varphi}{\eta\ \exists x\varphi}$$

5.2 Similar to 4.6(a).