Midterm Exam 2

Math 401

April 9, 2015

Please use your own notebook paper, and write in pencil if possible. Feel free to do the problems out of order.

1. (25 points) Consider the following commutative rings with four elements:

(i) $\mathbb{Z}_4$

(ii) $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) $\mathbb{Z}_2[x]/(x^2)$

(iv) $\mathbb{Z}_2[y]/(y^2+1)$

(v) $\mathbb{Z}_2[z]/(z^2+z)$

(vi) $\mathbb{Z}_2[w]/(w^2+w+1)$

Show that:

(a) Show that (ii) is isomorpic to (v).
Hint: Consider the map $\varphi\colon \mathbb{Z}_2[z] \to \mathbb{Z}_2 \times \mathbb{Z}_2$ defined by $\varphi(f) = (f(0), f(1))$.

(b) Show that (iii) is isomorphic to (iv).
Hint: Consider the map $\varphi\colon \mathbb{Z}_2[x] \to \mathbb{Z}_2[y]/(y^2+1)$ given by $\varphi(f) = \overline{f(y+1)}$. Observe that $y^2+1 = (y+1)^2$.

(c) Show that (i) is not isomorphic to any of the others.
Hint: In (i) we have $1+1 \neq 0$.

(d) Show that (ii) is not isomorpic to (iii).
Hint: In (ii) there is no non-zero element a with $a^2 = (0,0)$.

(e) Show that (vi) is not isomorphic to any of the others.
Hint: w^2+w+1 is irreducible, so (vi) is a field.

2. (0 points) I was going to ask you to show that if $f, g \in \mathbb{Q}[x]$ then their gcd in $\mathbb{Q}[x]$ is the same as their gcd in $\mathbb{R}[x]$, but the exam was getting too long. Maybe I'll put it on the final, but more likely I'll put something about the ring $\mathbb{Z}[\sqrt{-3}]$ and then a bunch of group theory.

3. (25 points) Let R be a commutative ring and $I, J \subset R$ two ideals. Recall that IJ is the ideal generated by elements of the form ij where $i \in I$ and $j \in J$:

$$IJ = \{i_1 j_1 + i_2 j_2 + \cdots + i_k j_k : \\ i_1, \dots, i_k \in I,\ j_1, \dots, j_k \in J,\ k \text{ is allowed to vary}\}.$$

(a) Show that $IJ \subset I \cap J$.

(b) Give an example with $R = \mathbb{Z}$ where IJ is a proper subset of $I \cap J$, that is, where they are not equal.

(c) Show that if $I + J = (1)$ then $IJ = I \cap J$.

(d) Show that the converse of part c holds when $R = \mathbb{Z}$ and $I, J \neq (0)$. (Hint: This isn't meant to be hard. Use the well-known description of the gcd and lcm of two integers in terms of their prime factorizations.)

(e) Show that the converse of part c does not hold in general by taking $R = \mathbb{Z}[x]$, $I = (2)$, and $J = (x)$.