Midterm Exam 1

Math 401

February 24, 2015

Please use your own notebook paper, and write in pencil if possible. Feel free to do the problems out of order.

1. (10 points)

(a) Let $f = x^6 - 1$ and $g = x^4 - 1$ in $\mathbb{Q}[x]$. Calculate $d = \gcd(f, g)$. Find polynomials $a, b \in \mathbb{Q}[x]$ such that $af + bg = d$.

(b) List all *complex* roots of f and g. What roots do they have in common? Does this agree with the gcd you found above?

2. (20 points)

(a) Show that $f = 8x^3 - 6x - 1$ is irreducible in $\mathbb{Z}[x]$. Hint: Either show that it has no root in $\mathbb{Q}$ using the rational root test, or reduce mod 5 and show that it has no root in $\mathbb{Z}_5$. Why is this enough?

(b) Calculate $f(-\frac{1}{2})$ and $f(\frac{1}{2})$. Conclude that f has three real roots.

(c) Find the three real roots of f. You may use Cardano's formula: The roots of $z^3 + pz + q = 0$ are given by $v - \frac{p}{3v}$, where

$$v = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}},$$

here you take one of the two square roots, but all three cube roots.

(Continued on next page.)

3. (25 points) Let R be an integral domain. Recall that an element $a \in R$ is called *irreducible* if it is not zero, not a unit, and $a = bc$ implies that b is a unit or c is a unit; and that a is called *prime* if $a \mid bc$ implies $a \mid b$ or $a \mid c$. In lecture we saw that if $a \neq 0$ and a is prime then a is irreducible. Here we will show the converse under some hypotheses.

(a) Let $a, b \in R$. Show that $a \mid b$ if and only if $(b) \subset (a)$.

(b) Let $a, b \in R$. Show that $a \mid b$ if and only if $(a, b) = (a)$.

(c) Show that $a \in R$ is a unit if and only if $(a) = R$.

Now suppose that R is a *principal ideal domain*, that is, an integral domain where every ideal is generated by one element; in particular, for every $a, b \in R$ there is a $c \in R$ such that $(a, b) = (c)$.

(d) Show that if $a \in R$ is irreducible then for every $b \in R$, either $(a, b) = (a)$ or $(a, b) = R$.

(e) Show that if $a \in R$ is irreducible then a is prime. Hint: Suppose that $a \mid bc$ and $a \nmid b$; first argue that you can write $1 = ax + by$; then multiply through by c.