Final Exam

Math 401

April 26, 2015

1. Recall that a group G is called *cyclic* if there is an element $g \in G$ such that every element of G is a power of g:

$$G = \{1, g^{\pm 1}, g^{\pm 2}, g^{\pm 3}, \dots\}.$$

(a) Show that if G is cyclic and H is a subgroup of G then H is cyclic. Hint: Let k be the least positive integer such that $g^k \in H$. If g^l is any other element of H then $k \mid l$; otherwise by long division we can write $l = qk + r$ for some $0 < r < k$, so...

(b) Consider a Rubik's cube. You might hope there is some clever sequence of moves such that if you take any mixed-up cube and make this sequence of moves over and over, you will eventually get to the "solved" position, no matter where you started. Show that this is not the case by observing that the symmetry group G of the Rubik's cube contains a copy of $\mathbb{Z}_4 \times \mathbb{Z}_4$, given by rotating the top and the bottom, and this is not a cyclic group.*

2. Let S_n denote the symmetric group on n letters.

(a) Show that an m-cycle $(a_1\, a_2\, \cdots\, a_m)$ is an even permutation if and only if m is odd. Hint: The point is just to remember how to factor σ as a product of transpositions.

(b) Let $A_n \subset S_n$ be the subgroup of even permutations; then the order of A_n is half that of S_n. Why?

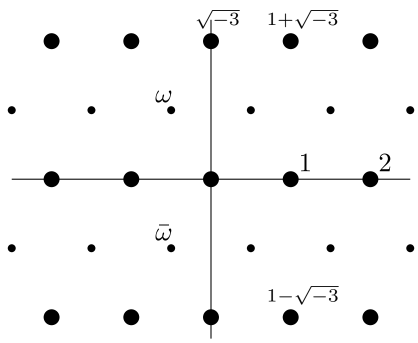
(c) List the 12 elements of A_4, in cycle notation.

(d) Let A_4 act on itself by conjugation. How many orbits are there? How many elements are in each orbit? Hint: To save time, it is enough to see how $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, and $(1\ 4)(2\ 3)$ act on the 3-cycles, and how $(1\ 2\ 3)$ acts on the products of 2-cycles; then note that conjugation preserves cycle type, and there is no orbit of order eight (why?), so the orbits can't get any bigger than what you've already found at this point. Further hint: If this is taking you forever, go work on another problem and maybe come back later.

*There is an algorithm for solving the Rubik's cube, but it is not as simple as "make this sequence of moves over and over."

3. Let R be a ring and S a subring. Show that if I is an ideal of R then $J = I \cap S$ is an ideal of S. Show that if I is a prime ideal of R then J is a prime ideal of S.

4. Let $S = \mathbb{Z}[\sqrt{-3}]$, and let $R = \mathbb{Z}[\omega]$, where $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$ is a cube root of unity. Observe that $\sqrt{-3} = 2\omega + 1$, so S is a subring of R, and that both are subrings of $\mathbb{C}$ and thus in particular integral domains. Here is a picture of R, with the subring S shown in heavier dots:



(a) Observe that if $z = a + b\sqrt{-3} \in S$ then $|z|^2 = a^2 + 3b^2$, which is a non-negative integer for all $a, b \in \mathbb{Z}$. Conclude that z is a unit in S if and only if $|z| = 1$. List the units in S. Hint: There are two.

(b) Observe that if $z = a + b\omega \in R$ then $|z|^2 = a^2 - ab + b^2$, which is a non-negative integer for all $a, b \in \mathbb{Z}$: non-negative because it's $|z|^2$, and an integer because it's $a^2 - ab + b^2$. Conclude that z is a unit in R if and only if $|z| = 1$. List the units in R. Hint: There are six.

(c) Show that 2 is irreducible in S. Hint: Argue that there is no $z \in S$ with $|z|^2 = 2$. Thus if $2 = zw$ then...

(d) Observe that $4 = 2 \cdot 2 = (1 + \sqrt{-3})(1 - \sqrt{-3})$. Conclude that 2 is not prime in S.

(e) I was going to ask you to show that 2 is prime (and hence irreducible) in R, but this turns out to be a pain.

(f) Show that the ideals generated by 2 and $1 - \sqrt{-3}$ in R are the same, but that the ideals they generate in S are different. Hint: Recall from homework that $(a) = (b)$ if and only if there is a unit u such that $a = ub$. Warning: Consider writing $(2)_R$ and $(2)_S$ rather than just (2) to avoid confusion.

(g) Let $I = (2)_R \subset R$ and $J = I \cap S$. Show that $J = (2, 1 - \sqrt{-3})_S$, and that this is not a principal ideal. Hint: It is clear that $2 \in I \cap S$ and $1 - \sqrt{-3} \in I \cap S$; for the reverse containment, note that a general element of I is of the form $2(a + b\omega)$, and multiply that out.