Solutions to Midterm Exam 1

Math 401

October 7, 2014

It is possible to do well even if you didn't finish.

1. (20 points) Show by induction on n that

$$1^4 + 2^4 + \cdots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

Solution: For $n = 1$ the claim is that $\frac{1}{5} + \frac{1}{2} + \frac{1}{3} - \frac{1}{30} = 1$, which is true. Suppose the claim holds for n; then we have

$$\begin{aligned}
&\frac{(n+1)^5}{5} + \frac{(n+1)^4}{2} + \frac{(n+1)^3}{3} - \frac{n+1}{30} \\
&\qquad = \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1}{5} \\
&\qquad\qquad + \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{2} + \frac{n^3 + 3n^2 + 3n + 1}{3} - \frac{n+1}{30} \\
&\qquad = \frac{1}{5}n^5 + \left(1 + \frac{1}{2}\right)n^4 + \left(4 + \frac{1}{3}\right)n^3 + 6n^2 + \left(4 - \frac{1}{30}\right)n^2 + 1 \\
&\qquad = \left(\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\right) + (n^4 + 4n^3 + 6n^2 + 4n + 1) \\
&\qquad = (1^4 + 2^4 + \cdots + n^4) + (n+1)^4,
\end{aligned}$$

so the claim holds for $n+1$. Alternatively, you might prefer to assume the claim for $n-1$ and prove it for n:

$$
\begin{aligned}
&1^4 + 2^4 + \cdots + (n-1)^4 + n^4 \\
&\quad = \left(\frac{(n-1)^5}{5} + \frac{(n-1)^4}{2} + \frac{(n-1)^3}{3} - \frac{n-1}{30} \right) + n^4 \\
&\quad = \Bigg(\frac{n^5 - 5n^4 + 10n^3 - 10n^2 + 5n - 1}{5} \\
&\qquad\quad + \frac{n^4 - 4n^3 + 6n^2 - 4n + 1}{2} + \frac{n^3 - 3n^2 + 3n - 1}{3} - \frac{n-1}{30} \Bigg) + n^4 \\
&\quad = \left(\frac{n^5}{5} - \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \right) + n^4 \\
&\quad = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.
\end{aligned}
$$

2. (10 points)

(a) Solve this system of congruences:

$$x \equiv 1 \pmod{2} \qquad x \equiv 2 \pmod{3} \qquad x \equiv 3 \pmod{5}$$

Solution: Rewriting the first two as $x \equiv -1 \pmod{2}$ and $x \equiv -1 \pmod{3}$ we see that they are equivalent to $x \equiv -1 \pmod{6}$. Now writing $1 = 6 - 5$ we see that $6 \cdot 3 - 5 \cdot (-1) = 23$ is a solution to all three, so

$$x \equiv 23 \pmod{30}$$

is the general solution.

(b) Which of these can be solved? Solve it.

(i) $x \equiv 5 \pmod{6}$ and $x \equiv 1 \pmod{9}$.
(ii) $x \equiv 5 \pmod{6}$ and $x \equiv 2 \pmod{9}$.
(iii) $x \equiv 5 \pmod{6}$ and $x \equiv 3 \pmod{9}$.

Solution: Observe that $\gcd(6, 9) = 3$, and that $x \equiv 5 \pmod{6}$ implies $x \equiv 2 \pmod{3}$. Now $x \equiv 1 \pmod{9}$ implies $x \equiv 1 \pmod{3}$, so (i) is impossible. Similarly, $x \equiv 3 \pmod{9}$ implies $x \equiv 0 \pmod{3}$, so (iii) is impossible. But (ii) can be solved, and the solution is

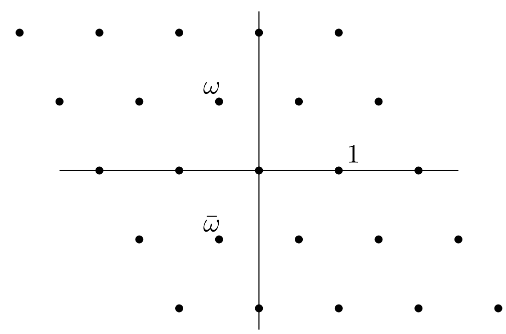
$$x \equiv 11 \pmod{18}.$$

3. (40 points) Let $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{3}i}{2}$ be the usual primitive cube root of unity. Recall that $\omega^2 = \omega^{-1} = \bar{\omega}$. This problem is about my favorite ring, the *Eisenstein integers*:

$$\mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\}.$$

(a) Roughly plot $a + b\omega$ on the complex plane for $-2 \leq a, b \leq 2$.

Solution:



(b) Show that $\bar{\omega} \in \mathbb{Z}[\omega]$. Deduce that if $z \in \mathbb{Z}[\omega]$ then $\bar{z} \in \mathbb{Z}[\omega]$. Label 1, ω, and $\bar{\omega}$ on your answer to part (a).

Solution: We have $\bar{\omega} = \frac{-1-\sqrt{3}i}{2} = -\omega - 1$. Thus if $z = a + b\omega$ then

$$\bar{z} = a + b\bar{\omega} = a + b(-\omega - 1) = (a - b) - b\omega \in \mathbb{Z}[\omega].$$

(c) Show that if $z \in \mathbb{Z}[\omega]$ then $|z|^2$ is a positive* integer.

Solution: We know that $|z|^2$ is positive* for any $z \in \mathbb{C}$, so we need only show that it is an integer for $z \in \mathbb{Z}[\omega]$. If $z = a + b\omega$ then

$$\begin{aligned} |z|^2 = z\bar{z} &= (a + b\omega)(a + b\bar{\omega}) \\ &= a^2 + ab(\omega + \bar{\omega}) + b^2\omega\bar{\omega} \\ &= a^2 - ab + b^2, \end{aligned}$$

which is an integer because a and b are integers.

(d) Show that an element $z \in \mathbb{Z}[\omega]$ is a unit if and only if $|z|^2 = 1$.

Solution: First suppose that $z \in \mathbb{Z}[\omega]$ is a unit. Then there is a $z' \in \mathbb{Z}[\omega]$ such that $zz' = 1$. Taking $|\cdot|^2$ on both sides we get $|z|^2|z'|^2 = 1$; but $|z|^2$ and $|z'|^2$ are positive integers, so $|z|^2 = |z'|^2 = 1$.

Conversely, suppose that $|z|^2 = 1$. Then $z\bar{z} = 1$, and we have seen that $\bar{z} \in \mathbb{Z}[\omega]$, so z is a unit.

*Correction 10/15/14: This should have been "non-negative."

(e) List all the units in $\mathbb{Z}[\omega]$. (Hint: Refer to your answer to part (a). There are finitely many.)

Solution: They are the six points lying on the unit circle, namely ± 1, $\pm\omega$, and $\pm\bar{\omega} = \mp 1 \mp \omega$. (In contrast, the ring $\mathbb{Z}[\sqrt{3}]$ has infinitely many units: $\pm(2+\sqrt{3})^n$ for any $n \in \mathbb{Z}$.)

(f) Are there any zero-divisors in $\mathbb{Z}[\omega]$?

Solution: No, because there are no zero-divisors in $\mathbb{C}$.

(g) Show that $z = 3 + 2\omega$ is not a unit, and if it can be factored as $z_1 z_2$ for some $z_1, z_2 \in \mathbb{Z}[\omega]$ then one of z_1, z_2 is a unit. Such an element is called *irreducible*.

Solution: We have $|z|^2 = 3^2 - 3 \cdot 2 + 2^2 = 7$, so z is not a unit. If $z = z_1 z_2$ then $|z|^2 = |z_1|^2 |z_2|^2$. Since $|z|^2 = 7$ is prime, we either have $|z_1|^2 = 1$ and $|z_2|^2 = 7$, so z_1 is a unit, or $|z_1|^2 = 7$ and $|z_2|^2 = 1$, so z_2 is a unit. (It is interesting to note that while 7 is prime in $\mathbb{Z}$, it factors as $z\bar{z} = (3+2\omega)(1-2\omega)$ in $\mathbb{Z}[\omega]$.)

4. (15 points) This problem is about Cardano's formula for the roots of the cubic polynomial

$$x^3 + px + q, \qquad\qquad p \neq 0.$$

Let

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \qquad\qquad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

be the roots of the quadratic polynomial

$$y^2 + qy - \frac{p^3}{27}.$$

(a) Find AB.

Solution: Since A and B are the roots of the quadratic above, AB equals its constant term $-\frac{p^3}{27}$.

(b) Show that for every v with $v^3 = A$ there is a w with $w^3 = B$ such that

$$w - \frac{p}{3w} = v - \frac{p}{3v}.$$

(Hint: $B = \frac{AB}{A}$.)

Solution: We want

$$w^3 = B = \frac{AB}{A} = \frac{-p^3/27}{v^3} = \left(-\frac{p}{3v}\right)^3,$$

so we take $w = -\frac{p}{3v}$. This implies that $v = -\frac{p}{3w}$, so

$$w - \frac{p}{3w} = -\frac{p}{3v} + v$$

as desired.

(The point was to show that either square root of $\frac{q^2}{4} + \frac{p^3}{27}$ suffices to give all three roots of $x^3 + px + q$.)

5. (15 points)

(a) Show that $f = x^2 + x - 1$ and $g = x^3 + x^2 - 1$ are irreducible in $\mathbb{Z}_3[x]$.

Solution: Since f is quadratic and g is a cubic, it is enough to show that they don't have any roots in $\mathbb{Z}_3$:

x	$f(x)$	$g(x)$
0	-1	-1
1	1	1
-1	-1	$-1.$

Note that $x^3 \equiv x \pmod 3$ by Fermat's little theorem, which explains why f and g take the same values.

(b) Find $a, b \in \mathbb{Z}_3[x]$ such that $af + bg = 1$.

Solution: Divide g by f to get a quotient x and a remainder $x - 1$, so

$$g = xf + (x - 1).$$

Divide f by $x - 1$ to get a quotient $x - 1$ and a remainder 1, so

$$f = (x - 1)(x - 1) + 1.$$

Rewrite the first as

$$(x - 1) = g - xf$$

and the second as

$$\begin{aligned} 1 &= f - (x - 1)(x - 1) \\ &= f - (x - 1)(g - xf) \\ &= (x^2 - x + 1)f - (x - 1)g, \end{aligned}$$

so we can take $a = x^2 - x + 1$ and $b = -x + 1$. There are many other solutions, so yours may be different, but this is the simplest.