Midterm Exam 1

Math 401

October 7, 2014

Please use your own notebook paper, and write in pencil if possible. Feel free to do the problems out of order.

1. (20 points) Show by induction on n that

$$1^4 + 2^4 + \cdots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

2. (10 points)

(a) Solve this system of congruences:

$$x \equiv 1 \pmod 2 \qquad x \equiv 2 \pmod 3 \qquad x \equiv 3 \pmod 5$$

(b) Which of these can be solved? Solve it.

(i) $x \equiv 5 \pmod 6$ and $x \equiv 1 \pmod 9$.
(ii) $x \equiv 5 \pmod 6$ and $x \equiv 2 \pmod 9$.
(iii) $x \equiv 5 \pmod 6$ and $x \equiv 3 \pmod 9$.

3. (40 points) Let $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{3}i}{2}$ be the usual primitive cube root of unity. Recall that $\omega^2 = \omega^{-1} = \bar{\omega}$. This problem is about my favorite ring, the *Eisenstein integers*:

$$\mathbb{Z}[\omega] = \{a + b\omega : a, b \in \mathbb{Z}\}.$$

(a) Roughly plot $a + b\omega$ on the complex plane for $-2 \leq a, b \leq 2$.

(b) Show that $\bar{\omega} \in \mathbb{Z}[\omega]$. Deduce that if $z \in \mathbb{Z}[\omega]$ then $\bar{z} \in \mathbb{Z}[\omega]$. Label 1, ω, and $\bar{\omega}$ on your answer to part (a).

(c) Show that if $z \in \mathbb{Z}[\omega]$ then $|z|^2$ is a positive* integer.

(d) Show that an element $z \in \mathbb{Z}[\omega]$ is a unit if and only if $|z|^2 = 1$.

(e) List all the units in $\mathbb{Z}[\omega]$. (Hint: Refer to your answer to part (a). There are finitely many.)

*Correction 10/15/14: This should have been "non-negative."

(f) Are there any zero-divisors in $\mathbb{Z}[\omega]$?

(g) Show that $z = 3 + 2\omega$ is not a unit, and if it can be factored as $z_1 z_2$ for some $z_1, z_2 \in \mathbb{Z}[\omega]$ then one of z_1, z_2 is a unit. Such an element is called *irreducible*.

4. (15 points) This problem is about Cardano's formula for the roots of the cubic polynomial

$$x^3 + px + q, \qquad\qquad p \neq 0.$$

Let

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \qquad\qquad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

be the roots of the quadratic polynomial

$$y^2 + qy - \frac{p^3}{27}.$$

(a) Find AB.

(b) Show that for every v with $v^3 = A$ there is a w with $w^3 = B$ such that

$$w - \frac{p}{3w} = v - \frac{p}{3v}.$$

(Hint: $B = \frac{AB}{A}$.)

(The point was to show that either square root of $\frac{q^2}{4} + \frac{p^3}{27}$ suffices to give all three roots of $x^3 + px + q$.)

5. (15 points)

(a) Show that $f = x^2 + x - 1$ and $g = x^3 + x^2 - 1$ are irreducible in $\mathbb{Z}_3[x]$.

(b) Find $a, b \in \mathbb{Z}_3[x]$ such that $af + bg = 1$.