Second Midterm

Name: ________________________________

April 2, 2013

1. (a) The lengths of the two sides of a triangle are x and y, and θ is the angle between the two sides. Draw a picture.

   (b) The area of the triangle is $A = \frac{1}{2}xy\sin\theta$. If x is increasing at a rate of 2 inches per second, y is decreasing at a rate of 2 inches per second and θ is increasing at a rate of 0.1 radians per second, how fast is the area changing at the instant when $x = y = 3$ feet, and $\theta = \pi/3$ radians?

2. Write an equation for the tangent plane to the surface

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3$$

at the point $(2, -2, -1)$. Hint: Don't waste time multiplying out the right-hand side.

3. Let $f(x, y) = x(x^2 + y^2 - 1)$.

   (a) Sketch the level set $f(x, y) = 0$.

   (b) Find $f(2, 0)$. From this piece of information and your sketch above, guess where the function is positive and negative; indicate it with plusses and minuses on your sketch.

(c) Find the four critical points of f. Add them to your sketch.

(d) Decide whether each critical point is a local minimum, a local maximum, or a saddle point.

(e) Find the maximum value of f on the circle $x^2+y^2=4$ using Lagrange multipliers.

(f) Find the maximum value of f on the disc $x^2+y^2 \leq 4$. Does it occur in the interior or on the boundary (or both)?

(g) In what direction does f increase most steeply at the origin? At the point $(0,2)$? Add the appropriate vectors to your sketch.

4. (a) Sketch the surface $z = 2 - x^2 - y^2$.

(b) Find the volume of the region that is below the surface and above the xy-plane. Do the integral in polar coordinates.

5. This problem is about Jacobians (section 14.9).

(a) Sketch the portions of the curves $y = x^2$, $y = x^2 + 1$, $y = 2 - x^2$, and $y = 3 - x^2$ that lie in the first quadrant. Shade the curvy rectangle that they bound and label it R.

(b) We wish to find the center of mass of this region R. You can find the y-coordinate by symmetry, without doing any integrals. What is it?

(c) Let $u = y + x^2$ and $v = y - x^2$. What do the four curves in part (a) become in these new coordinates? Indicate this on your sketch, or draw a new one if the old one is becoming a mess.

(d) Solve for x and y in terms of u and v. Hint: Consider $u+v$ and $u-v$.

(e) Evaluate the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}.$$

Don't be discouraged by the profusion of twos.

(f) When you take an integral

$$\iint_R \text{something } dx\, dy$$

and change variables to u and v, what will happen to the bounds? What will happen to $dx\, dy$?

(g) Find $\iint_R x \; dx\, dy$. Hint: There's a lot of cancellation.

(h) Find the area of the region, i.e. find $\iint_R 1\,dx\,dy$.
Hint: Notice that $\int (u-v)^{-1/2}\,dv = -2(u-v)^{1/2} + C$. The integral is doable but the answer isn't very pretty.

(i) Find the x-coordinate of the center of mass. You don't need to simplify it.