Second Midterm

Name: **Solutions**

November 13, 2013

1. (20 points) Find the maximum and minimum of the function $f(x, y) = x^2y^2$ subject to the constraint $x^2 + y^2 = 3$. Hint: If you divide through by x, also consider the possibility that $x = 0$.
Solution: We let $g(x, y) = x^2 + y^2$, and solve $\nabla f = \lambda \nabla g$ and $g = 3$, that is,

$$2xy^2 = 2\lambda x \qquad 2x^2y = 2\lambda y \qquad x^2 + y^2 = 3.$$

From the first equation we see that $x = 0$ or $y^2 = \lambda$. From the second we see that $y = 0$ or $x^2 = \lambda$. Thus there are four cases to consider.

Case 1: $x = 0$, $y = 0$. In this case the constraint $x^2 + y^2 = 3$ cannot be satisfied.

Case 2: $x = 0$, $x^2 = \lambda$. We can ignore the fact that $x^2 = \lambda$; the constraint $x^2 + y^2 = 3$ gives $y = \pm\sqrt{3}$, and we have $f(0, \pm\sqrt{3}) = 0$.

Case 3: $y^2 = \lambda$, $y = 0$. This is similar to the previous case, and we get $f(\pm\sqrt{3}, 0) = 0$.

Case 4: $y^2 = \lambda$, $x^2 = \lambda$. The constraint $x^2 + y^2 = 3$ becomes $\lambda + \lambda = 3$, so $\lambda = 3/2$. Then $f(x, y) = x^2y^2 = \lambda^2 = 9/4$.

Thus the minimum value of the function is 0, and the maximum is 9/4.

2. (a) (10 points) Find the critical points of the function $f(x, y) = (\sin x)(2 + \sin y)$ in the region $0 \le x < 2\pi$, $0 \le y < 2\pi$. Hint: There are four.
Solution: We have

$$f_x(x, y) = (\cos x)(2 + \sin y) \qquad f_y(x, y) = \sin x \cos y.$$

If $f_x = 0$ then either $\cos x = 0$, so $x = \pi/2$ or $3\pi/2$; or else $(2 + \sin y) = 0$, so $\sin y = -2$ which is impossible. If $f_y = 0$ then either $\sin x = 0$, which is impossible since we just found that $x = \pi/2$ or $3\pi/2$; or else $\cos y = 0$, so $y = \pi/2$ or $3\pi/2$. Thus the four critical points are

$$(\pi/2, \pi/2) \qquad (\pi/2, 3\pi/2) \qquad (3\pi/2, \pi/2) \qquad (3\pi/2, 3\pi/2).$$

(b) (10 points) For each critical point, decide whether it is a local maximum, a local minimum, or a saddle point. Hint: Each occurs at least once.
Solution: We have

$$f_{xx}(x,y) = -(\sin x)(2+\sin y) \quad f_{xy}(x,y) = \cos x \cos y \quad f_{yy}(x,y) = -\sin x \sin y.$$

Substituting the four points above into the Hessian matrix

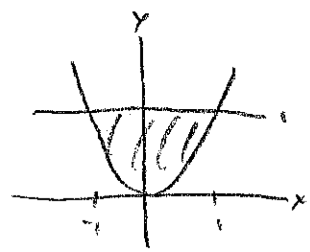
$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

we get

$$\begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the first case we have a local maximum, in the second a saddle point, in the third a local minimum, and in the fourth another saddle point.

3. (a) (5 points) Sketch the plane region lying above the parabola $y = x^2$ and below the line $y = 1$.
Solution:



(b) (5 points) Find the area of the region.
Solution:

$$\int_{-1}^{1} (1-x^2)\,dx = 2\int_0^1 (1-x^2)\,dx = 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}.$$

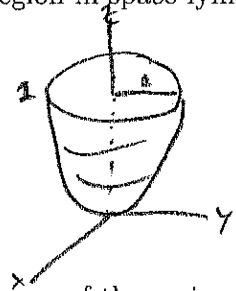
(c) (10 points) Find the centroid of the region. Hint: For the y-coordinate you need to do a double integral, but for the x-coordinate you don't.
Solution: The x-coordinate is zero. For the y-coordinate, we have

$$\begin{aligned}\int_{-1}^{1}\int_{x^2}^{1} y\,dy\,dx &= 2\int_0^1\int_{x^2}^{1} y\,dy\,dx \\ &= \int_0^1 \left[y^2\right]_{x^2}^{1} dx = \int_0^1 (1-x^4)\,dx = \left[x - \frac{x^5}{5}\right]_0^1 = \frac{4}{5}.\end{aligned}$$

Since $\frac{4/5}{4/3} = 3/5$, the centroid is at $(0, \frac{3}{5})$.

4. (a) (5 points) Sketch the region in space lying above the paraboloid $z = x^2 + y^2$ and below the plane $z = 1$.

Solution:



(b) (5 points) Find the volume of the region. Hint: The thing is round. If you want to use rectangular coordinates that's your business, but if you do I won't give partial credit.

Solution: In polar or cylindreical coordinates the paraboloid $z = x^2+y^2$ becomes $z = r^2$. We have

$$\int_0^{2\pi}\int_0^1 (1-r^2)\, r\, dr\, d\theta = 2\pi \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = \frac{\pi}{2}.$$

(c) (10 points) Find the centroid of the region. Hint: For the z-coordinate you need to do a triple integral, but for the x- and y-coordinates you don't.

Solution: The x- and y-coordinates are zero. For the z-coordinate, we have

$$\begin{aligned}\int_0^{2\pi}\int_0^1\int_{r^2}^1 z\, r\, dz\, dr\, d\theta &= 2\pi \int_0^1\int_{r^2}^1 zr\, dz\, dr \\ &= \pi \int_0^1 \left[z^2 r\right]_{r^2}^1 dr = \pi \int_0^1 (r - r^5)\, dr = \pi \left[\frac{r^2}{2} - \frac{r^6}{6}\right]_0^1 = \frac{\pi}{3}.\end{aligned}$$

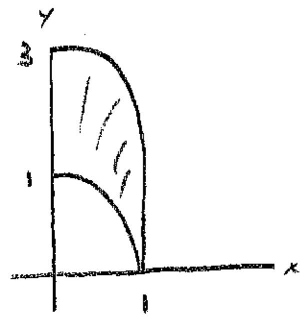
Since $\frac{\pi/3}{\pi/2} = 2/3$, the centroid is at $(0, 0, 2/3)$.

(d) (moral points) Is the centroid of this region higher or lower than that of the plane region in the previous problem? Is that reasonable?

Solution: It's higher: $2/3 = 0.66\ldots$ is greater than $3/5 = 0.6$. This is reasonable since by spinning the plane region around we've added more mass toward the top than toward the bottom.

5. (a) (5 points) Sketch the region R in the first quadrant that lies between the curves $y = 1 - x^2$ and $y = 3(1 - x^2)$.

Solution:



(b) (5 points) Consider the change of coordinates $x = u, \quad y = v(1 - u^2)$. What do the region's bounds become in these coordinates?

Solution: $0 \leq u \leq 1, \quad 1 \leq v \leq 3.$

(c) (5 points) Find the Jacobian $\dfrac{\partial(x, y)}{\partial(u, v)}$.

Solution: We have

$$x_u = 1 \qquad\qquad x_v = 0$$
$$y_u = -2uv \qquad\qquad y_v = 1 - u^2.$$

Thus

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = 1 - u^2.$$

(d) (5 points) Evaluate $\displaystyle\iint_R \frac{x^2}{y}\, dx\, dy$ using this change of coordinates.

Solution:

$$\iint_R \frac{x^2}{y}\, dx\, dy = \int_1^3 \int_0^1 \frac{u^2}{v(1 - u^2)}\, (1 - u^2), du\, dv = \int_1^3 \int_0^1 u^2 v^{-1}\, du\, dv$$
$$= \int_1^3 v^{-1}\, dv \cdot \int_0^1 u^2 du = \Big[\ln v\Big]_1^3 \cdot \left[\frac{u^3}{3}\right]_0^1 = \ln 3 \cdot \frac{1}{3}.$$