Messy Lagrange Multipliers Example

Math 212

Thursday, September 27, 2012

Problem: Find the points on the ellipse $x^2 + xy + 2y^2 = 1$ farthest from the origin using the method of Lagrange multipliers.

Solution: We maximize the function $f(x, y) = x^2 + y^2$ with the constraint $g(x, y) = x^2+xy+2y^2 = 1$. We have $\nabla f = \langle 2x, 2y\rangle$ and $\nabla g = \langle 2x+y, x+4y\rangle$, so we want to solve the equations

$$2x = \lambda(2x + y) \tag{1}$$
$$2y = \lambda(x + 4y) \tag{2}$$
$$x^2 + xy + 2y^2 = 1. \tag{3}$$

First observe that for any given λ, equations (1) and (2) describe two lines through the origin: solving both for y we have

$$y = \frac{2 - 2\lambda}{\lambda}\, x \qquad\qquad y = \frac{\lambda}{2 - 4\lambda}\, x. \tag{4}$$

If these lines have different slopes then they intersect only at the origin, which contradicts (3), so they must have the same slope:

$$\frac{2 - 2\lambda}{\lambda} = \frac{\lambda}{2 - 4\lambda}.$$

Clearing denominators we have

$$4 - 12\lambda + 8\lambda^2 = \lambda^2$$
$$7\lambda^2 - 12\lambda + 2 = 0$$
$$\lambda = \frac{6 \pm 2\sqrt{2}}{7}. \tag{5}$$

Note that we found this on Thursday in a more complicated way.

We can use Sharrin/Andrew/Euler's trick that we saw in the $xyz = 8$ problem on Thursday to rule out one of the roots (5); this is not strictly necessary, but it cuts a messy calculation in half. Multiply equation (1) by x and equation (2) by y and add them up to get

$$2x^2 + 2y^2 = \lambda(2x^2 + 2xy + 4y^2).$$

Divide through by 2 and use equation (3) to get

$$x^2 + y^2 = \lambda.$$

Thus since we want the points *farthest* from the origin, we only want the larger root

$$\lambda = \frac{6 + 2\sqrt{2}}{7}.$$

Now the two lines (4) become

$$y = (1 - \sqrt{2})x.$$

Plugging this into (3) we get

$$(8 - 5\sqrt{2})x^2 = 1,$$

so

$$x = \pm\sqrt{\frac{8 + 5\sqrt{2}}{14}},$$

and thus

$$y = (1 - \sqrt{2})x = \mp\sqrt{\frac{4 - \sqrt{2}}{14}}.$$

Note that the plus-or-minus became minus-or-plus because $1 - \sqrt{2}$ is negative.