Solutions to Midterm Exam

Math 637

Friday, October 29, 2021

1. (a) Suppose that a Lie group G acts on manifolds M and N on the left, and that $F\colon M \to N$ is an equivariant map, meaning that $F(g \cdot p) = g \cdot F(p)$ for all $g \in G$ and all $p \in M$. Prove that if G acts transitively on M then F has constant rank.

   Solution: Let $p, q \in M$. Because G acts transitively on M, we can write $q = g \cdot p$ for some $g \in G$. Let $\varphi\colon M \to M$ be the map $\varphi(x) = g \cdot x$, and let $\psi\colon N \to N$ be defined similarly. We have $\varphi(p) = q$, and because F is equivariant, we have $F \circ \varphi = \psi \circ F$, so by the chain rule,

   $$DF_q \circ D\varphi_p = D\psi_{F(p)} \circ DF_p.$$

   We see that φ and ψ are diffeomorphisms, with inverses given by $x \mapsto g^{-1} \cdot x$, so $D\varphi_p$ and $D\psi_{F(p)}$ are isomorphisms, so DF_p has the same rank as DF_q.

   (b) Let $G = \mathrm{GL}_n(\mathbb{R})$, and let $F\colon G \to G$ be the map $F(A) = AA^\top$. Prove that F is not a group homomorphism.

   Solution: If F were a homomorphism then for all $A, B \in \mathrm{GL}_n(\mathbb{R})$ we would have $F(AB) = F(A)F(B)$, that is,

   $$ABB^\top A^\top = AA^\top BB^\top.$$

   It is enough to find an A and B for which this is not true. If $n = 2$, I will take

   $$A = B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

   Then we find that

   $$B^\top A^\top AB = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \qquad A^\top AB^\top B = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}.$$

   If $n > 2$ then I will take that 2×2 block and extend it with 1's down the diagonal.

(c) Let G act on the domain of F by left multiplication. Describe a different action of G on the codomain that makes F into an equivariant map.

Solution: Let G act on the codomain of F by $A \cdot X = AXA^\top$. This is a group action because

$$A \cdot B \cdot X = ABXB^\top A^\top = (AB)X(AB)^\top = (AB) \cdot X,$$

and it is smooth because it is defined by polynomials. Then we see that F is equivariant:

$$F(A \cdot X) = (AX)(AX)^\top = AXX^\top A^\top = A \cdot F(X).$$

(d) Conclude that the orthogonal group $\mathrm{O}(n) \subset G$ is a manifold.

(Hint: Lee's Theorem 5.12 states that if a map has constant rank then its fibers are submanifolds. You may use this without proof.)

Solution: Note that the action of G on itself by left multiplication is transitive, so F has constant rank by part (a), so its fibers are manifolds. Then $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{R})$ is the set of matrices satisfying $A^\top A = 1$, or equivalently $A^\top = A^{-1}$, or equivalently $AA^\top = 1$: thus $\mathrm{O}(n) = F^{-1}(1)$.

2. Let G be a Lie group, and let $G_0 \subset G$ be the connected component of the identity. Because manifolds are locally path connected, connected components are the same as path components, which might be easier to work with.

   (a) Prove that G_0 is a normal subgroup of G.

   Solution: First we prove that G_0 is a subgroup: given $a, b \in G_0$, we want to prove that $ab^{-1} \in G_0$. Choose a continuous path $\gamma\colon [0,1] \to G$ with $\gamma(0) = 1$ and $\gamma(1) = a$, and a continuous path δ from 1 to b. Then $t \mapsto \gamma(t) \cdot \delta(t)^{-1}$ is a continuous path from $1 \cdot 1^{-1} = 1$ to $a \cdot b^{-1}$, so $ab^{-1} \in G_0$ as desired.

   Next we prove that G_0 is normal: given $a \in G_0$ and $g \in G$, we want to prove that $gag^{-1} \in G_0$. Choose a continuous path γ from 1 to a. Then $t \mapsto g \cdot \gamma(t) \cdot g^{-1}$ is a continuous path from $g \cdot 1 \cdot g^{-1} = 1$ to $g \cdot a \cdot g^{-1}$, so $gag^{-1} \in G_0$ as desired.

   (b) Prove that any other connected component of G is diffeomorphic to G_0.

   Solution: Let K be such a component, let $p \in K$, and let $\varphi\colon G \to G$ be the diffeomorphism $\varphi(x) = p \cdot x$. The φ takes path components to path components, and $\varphi(1) = p$, so φ takes G_0 to K.

   (c) I was going to ask you to prove that if $U \subset G$ is a connected neighborhood of the identity, then the subgroup generated by U is G_0. But I think the exam is already long enough.