Midterm Exam

Math 637

Friday, October 29, 2021

All manifolds, maps, actions, etc. are smooth.

1. (a) Suppose that a Lie group G acts on manifolds M and N on the left, and that $F\colon M \to N$ is an equivariant map, meaning that $F(g \cdot p) = g \cdot F(p)$ for all $g \in G$ and all $p \in M$. Prove that if G acts transitively on M then F has constant rank.

   (b) Let $G = \mathrm{GL}_n(\mathbb{R})$, and let $F\colon G \to G$ be the map $F(A) = AA^\top$. Prove that F is not a group homomorphism.

   (c) Let G act on the domain of F by left multiplication. Describe a different action of G on the codomain that makes F into an equivariant map.

   (d) Conclude that the orthogonal group $\mathrm{O}(n) \subset G$ is a manifold.

   (Hint: Lee's Theorem 5.12 states that if a map has constant rank then its fibers are submanifolds. You may use this without proof.)

2. Let G be a Lie group, and let $G_0 \subset G$ be the connected component of the identity. Because manifolds are locally path connected, connected components are the same as path components, which might be easier to work with.

   (a) Prove that G_0 is a normal subgroup of G.

   (b) Prove that any other connected component of G is diffeomorphic to G_0.

   (c) I was going to ask you to prove that if $U \subset G$ is a connected neighborhood of the identity, then the subgroup generated by U is G_0. But I think the exam is already long enough.