Worksheet 8

Math 634, Algebraic Topology

Friday, October 16, 2020

Finish the proof of the Borsuk–Ulam theorem. Recall that we're given a continuous map $f\colon S^2 \to \mathbb{R}^2$, and we want to show that there's a point $x \in S^2$ with $f(x) = f(-x)$. If not, then the map $g\colon S^2 \to S^1$ given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|},$$

is well-defined. Let $h\colon S^1 \to S^1$ be the restriction of g to the equator.

1. Observe that h is an odd map, that is, $h(-z) = -h(z)$. Consider the loop $\gamma : I \to S^1$ given by

   $$\gamma(t) = h(e^{2\pi i t})/h(1).$$

   Show that $\varphi([\gamma])$ is an odd number, where $\varphi\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is the map we've been studying all week.

   Hint: First observe that $\tilde{\gamma}(\frac{1}{2}) = n + \frac{1}{2}$ for some $n \in \mathbb{Z}$. Then argue that for $t \in [\frac{1}{2}, 1]$ we have $\tilde{\gamma}(t) = \tilde{\gamma}(t - \frac{1}{2}) + n + \frac{1}{2}$, so $\gamma(1) = 2n + 1$.

2. But h extends to a map $D^2 \to S^1$, so $\varphi([\gamma]) = 0$ by §1.1 #5.

   Hint: The restriction of g to the the northern hemisphere.