Worksheet 6

Math 634, Algebraic Topology

Monday, October 12, 2020

In lecture, we started to prove the following "path lifting" statement: let $p\colon \mathbb{R} \to S^1$ be the map $p(x) = e^{2\pi i x}$, and let $\gamma\colon I \to S^1$ be a map with $\gamma(0) = 1$; then there is a unique map $\tilde{\gamma}\colon I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$ and $p \circ \tilde{\gamma} = \gamma$.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, dotted, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1. \end{tikzcd}$$

Work through the following outline of the proof, convincing yourselves that each claim is correct.

1. Let $U = S^1 \setminus \{-1\}$ and $V = S^1 \setminus \{1\}$. Draw a picture.

2. There is an integer $k \gg 0$ such that if we subdivide I into k equal intervals
$$[0, \tfrac{1}{k}],\ [\tfrac{1}{k}, \tfrac{2}{k}],\ \ldots,\ [\tfrac{k-1}{k}, 1]$$
then γ of each one is contained in U, or in V, or both.

   Sketch. The preimage $\gamma^{-1}(U) \subset I$ is a (possibly infinite) union of open intervals. So is $\gamma^{-1}(V)$. Together, these intervals cover I. Because I is compact, we can choose a finite subcollection that still cover I. Choose k big enough that each $[\frac{i-1}{k}, \frac{i}{k}]$ fits inside one of the open intervals in that finite subcollection.

3. Now $\gamma([0, \frac{1}{k}]) \subset U$, so there is a unique map
$$\tilde{\gamma}_1\colon [0, \tfrac{1}{k}] \to \mathbb{R}$$
with $\tilde{\gamma}_1(0) = 0$ and $p \circ \tilde{\gamma}_1 = \gamma|_{[0, \frac{1}{k}]}$.

   (Continues on next page.)

4. Next, $\gamma([\frac{1}{k}, \frac{2}{k}]) \subset U$ or V, so there is a unique map
$$\tilde{\gamma}_2 \colon [\tfrac{1}{k}, \tfrac{2}{k}] \to \mathbb{R}$$
with $\tilde{\gamma}_2(\frac{1}{k}) = \tilde{\gamma}_1(\frac{1}{k})$ and $p \circ \tilde{\gamma}_2 = \gamma|_{[\frac{1}{k}, \frac{2}{k}]}$.

5. Continuing in this way, we can lift on each interval $[\frac{i-1}{k}, \frac{i}{k}]$. Because the liftings agree on the endpoints $\frac{i}{k}$, they patch together to give a map
$$\tilde{\gamma} \colon I \to \mathbb{R}$$
with $\tilde{\gamma}(0) = 0$ and $p \circ \gamma = \gamma$, as desired.

6. In the course of the construction we saw that $\tilde{\gamma}$ was unique.