Worksheet 4

Math 634, Algebraic Topology

Wednesday, October 7, 2020

0. Introduce yourself to your colleague. Do they play any musical instruments?

1. Show that a space X is contractible if and only if the identity map $1\colon X \to X$ is nullhomotopic.

2. Show that S^∞ is contractible. More precisely, let

$$\begin{aligned}\mathbb{R}^\infty &= \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \cdots \\ &= \{(x_0, x_1, x_2, \dots) : x_i \in \mathbb{R} \text{ for all } i, \\ &\qquad\qquad \text{and the } x_i\text{s are eventually zero.}\},\end{aligned}$$

and define

$$S^\infty = \{(x_0, x_1, x_2, \dots) \in \mathbb{R}^\infty : \textstyle\sum x_i^2 = 1\}.$$

The inclusion

$$S^\infty \hookrightarrow \mathbb{R}^\infty \setminus 0$$

is a homotopy equivalence, just as in the finite-dimensional case, so it is enough to show that $\mathbb{R}^\infty \setminus 0$ is contractible. Show that the identity map of $\mathbb{R}^\infty \setminus 0$ is nullhomotopic, as follows:

   (a) Define a map $f\colon \mathbb{R}^\infty \setminus 0 \to \mathbb{R}^\infty \setminus 0$ by

   $$f(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots),$$

   shifting everything one place to the right. Show that $f \simeq 1$ via the straight-line homotopy.

   (b) Let $c\colon \mathbb{R}^\infty \setminus 0 \to \mathbb{R}^\infty \setminus 0$ be the constant map $c(x) = (1, 0, 0, 0, \dots)$. Show that $f \simeq c$ via the straight-line homotopy.