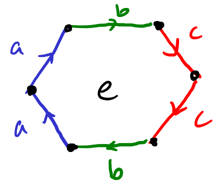
Worksheet 21

Math 634, Algebraic Topology

Wednesday, December 2, 2020

Let X be the surface determined by the diagram



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The boundary of the diagram gives a copy of $S^1 \vee S^1 \vee S^1 \subset X$, and the quotient $X/(S^1 \vee S^1 \vee S^1)$ is S^2. Thus the long exact sequence of the pair

$$\begin{array}{l}
0 \longrightarrow \tilde{H}_2(S^1 \vee S^1 \vee S^1) \longrightarrow \tilde{H}_2(X) \longrightarrow H_2(X, S^1 \vee S^1 \vee S^1) \\
\longrightarrow \tilde{H}_1(S^1 \vee S^1 \vee S^1) \longrightarrow \tilde{H}_1(X) \longrightarrow H_1(X, S^1 \vee S^1 \vee S^1) \\
\longrightarrow \tilde{H}_0(S^1 \vee S^1 \vee S^1) \longrightarrow \tilde{H}_0(X) \longrightarrow H_0(X, S^1 \vee S^1 \vee S^1) \longrightarrow 0
\end{array}$$

reads as

$$\begin{array}{l}
0 \longrightarrow 0 \longrightarrow \tilde{H}_2(X) \longrightarrow \mathbb{Z} \\
\longrightarrow \mathbb{Z}^3 \longrightarrow \tilde{H}_1(X) \longrightarrow 0 \\
\longrightarrow 0 \longrightarrow \tilde{H}_0(X) \longrightarrow 0 \longrightarrow 0,
\end{array}$$

and the map $\mathbb{Z} \to \mathbb{Z}^3$ can be described as $e \mapsto 2a + 2b + 2c$. Find the kernel and cokernel of this map, and hence $H_*(X)$.

[Hint: It might be helpful to choose a different basis for the $\mathbb{Z}^3$: rather than a, b, and c, you could take a, b, and $a + b + c$. But convince yourselves that this is a legitimate thing to do.]

Also find $H_*(X)$ with coefficients in $\mathbb{Q}$ and $\mathbb{Z}_2$.

If you have time to kill, convince yourselves that X is $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$, which is the same as (Klein bottle)$\#\mathbb{RP}^2$ and also (torus)$\#\mathbb{RP}^2$.