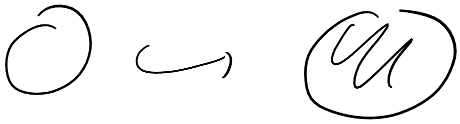
Solutions to Midterm 1

1. (a) Give an example of path-connected spaces X and Y and a map $f\colon X \to Y$ which is injective but not surjective, for which the induced map $f_*\colon \pi_1(X,x) \to \pi_1(Y, f(x))$ is injective but not surjective for some (and hence every) $x \in X$.

   Solution: Of course many answers are possible. One is to let f be the inclusion of the circle $S^1 \times \{1\}$ in the torus $S^1 \times S^1$. Then $f_*\colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is the map $n \mapsto (n, 0)$.

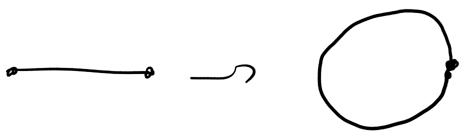
   (b) Give an example where f is injective but not surjective and f_* is surjective but not injective.

   Let $f\colon S^1 \to D^2$ be the inclusion of the boundary circle. Then $f_*\colon \mathbb{Z} \to 0$.



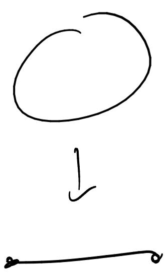
   (c) Give an example where f is surjective but not injective and f_* is injective but not surjective.

   Solution: Let $f\colon I \to S^1$ be given by $f(t) = e^{2\pi i t}$, which is not injective because $f(1) = f(0)$. Then $f_*\colon 0 \to \mathbb{Z}$.



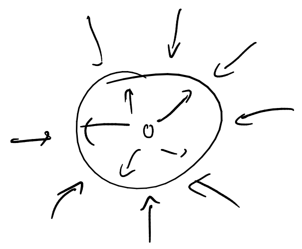
   (d) Give an example where f is surjective but not injective and f_* is surjective but not injective.

   Solution: Let $f\colon S^1 \to [-1, 1]$ be given by $f(x, y) = x$. Then $f_*\colon \mathbb{Z} \to 0$.



(e) Give one more example along these lines that you think is interesting.

Solution: Let $f\colon \mathbb{R}^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$ be given by $f(x) = x/|x|$. Then f is neither injective nor surjective, but $f_*\colon \mathbb{Z} \to \mathbb{Z}$ is the identity.



2. For a map $f\colon X \to Y$, the *mapping cylinder* M_f is defined on page 2 of Hatcher's book. I claim that if f is homotopic to another map $g\colon X \to Y$, then M_f and M_g are homotopy equivalent rel. $X \times \{0\}$.

(a) Write down explicit maps $\Phi\colon M_g \to M_f$ and $\Psi\colon M_f \to M_g$ that will turn out to be homotopy inverse to one another rel. $X \times \{0\}$. Be clear about why your maps are well-defined, but don't belabor the point.

Solution: Let $F\colon X \times I \to Y$ be a homotopy from f to g. Let $\Phi\colon M_g \to M_f$ be the map determined by

$$\Phi(x,s) = \begin{cases} (x, 2s) \in X \times I & \text{for } 0 \le s \le \frac{1}{2} \\ F(x, 2s-1) \in Y & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$
$$\Phi(y) = y.$$

The first rule gives a well-defined map $X \times I \to M_f$, because at $s = \frac{1}{2}$ we see that $(x,1) \in X \times I$ and $F(x,0) = f(x) \in Y$ map to the same point of M_f. The two rules together determine a map $(X \times I) \amalg Y \to M_f$, which descends to a map $M_g \to M_f$ because $\Phi(x,1) = F(x,1) = g(x) = \Phi(g(x))$.

Similarly, let $\Psi\colon M_f \to M_g$ be the map determined by

$$\Psi(x,s) = \begin{cases} (x, 2s) \in X \times I & \text{for } 0 \le s \le \frac{1}{2} \\ F(x, 2-2s) \in Y & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$
$$\Psi(y) = y.$$

(b) Write out the composition $\Phi \circ \Psi$ explicitly. Assert that it is homotopic to the identity, but don't write an explicit homotopy.

Solution: The composition $\Phi \circ \Psi \colon M_f \to M_f$ is determined by

$$\Phi(\Psi(x,s)) = \begin{cases} (x, 4s) \in X \times I & \text{for } 0 \le s \le \frac{1}{4} \\ F(x, 4s-1) \in Y & \text{for } \frac{1}{4} \le s \le \frac{1}{2} \\ F(x, 2-2s) \in Y & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$
$$\Phi(\Psi(y)) = y.$$

I asked you not to write out the homotopy, but I'll do it here in case you wanted to see it. First, the homotopy $M_f \times I \to M_f$ determined by

$$(x,s,t) \mapsto \begin{cases} (x, 4s) \in X \times I & \text{for } 0 \le s \le \frac{1}{4} \\ F(x, (1-t)(4s-1)) \in Y & \text{for } \frac{1}{4} \le s \le \frac{1}{2} \\ F(x, (1-t)(2-2s)) \in Y & \text{for } \frac{1}{2} \le s \le 1 \end{cases}$$
$$(y,t) \mapsto y$$

turns $\Phi \circ \Psi$ into

$$(x,s) \mapsto \begin{cases} (x, 4s) \in X \times I & \text{for } 0 \le s \le \frac{1}{4} \\ f(x) \in Y & \text{for } \frac{1}{4} \le s \le 1 \end{cases}$$
$$y \mapsto y,$$

which is the same as

$$(x,s) \mapsto \begin{cases} (x, 4s) \in X \times I & \text{for } 0 \le s \le \frac{1}{4} \\ (x, 1) \in X \times I & \text{for } \frac{1}{4} \le s \le 1 \end{cases}$$
$$y \mapsto y.$$

Then the homotopy

$$(x,s,t) \mapsto \begin{cases} (x, (1-t)\cdot 4s + ts) \in X \times I & \text{for } 0 \le s \le \frac{1}{4} \\ (x, (1-t)\cdot 1 + ts) \in X \times I & \text{for } \frac{1}{4} \le s \le 1 \end{cases}$$
$$(y,t) \mapsto y$$

turns this into the identity on M_f.

Similarly, $\Psi \circ \Phi$ is homotopic to the identity on M_g.

Sorry, I meant to ask for \Phi \circ \Psi, although it doesn't make much difference...

(c) If X is a point, then a homotopy from f to g is the same as a path from $f(x)$ to $g(x)$. Draw a picture of what Φ, Ψ, and $\Psi \circ \Phi$ look like in that case, to make it plausible that $\Psi \circ \Phi$ is homotopic to the identity. (You may want to do this part first.)