Solutions to Homework 6

§2.1 #12. *Show that chain homotopy of chain maps is an equivalence relation.*

Let $A_\bullet$ and $B_\bullet$ be two chain complexes, and let $f_\bullet$ be a chain map:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & A_i \arrow[r, "\partial_i"] \arrow[d, "f_i"] & A_{i-1} \arrow[r] \arrow[d, "f_{i-1}"] & \cdots \\
\cdots \arrow[r] & B_{i+1} \arrow[r, "\partial_{i+1}"] & B_i \arrow[r, "\partial_i"] & B_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

Reflexive: The zero map is a chain homotopy from $f_\bullet$ to $f_\bullet$: that is, if we take $P_i = 0\colon A_i \to B_{i+1}$, then $f_i - f_i = d_{i+1} \circ P_i + P_{i-1} \circ d_i$, so $f_\bullet \simeq f_\bullet$.

Symmetric: If $P_\bullet$ is a chain homotopy from $f_\bullet$ to another chain map $g_\bullet$, then $-P_\bullet$ is a chain homotopy from $g_\bullet$ to $f_\bullet$: that is, given maps $P_i\colon A_i \to B_{i+1}$ with $f_i - g_i = d_{i+1} \circ P_i + P_{i-1} \circ d_i$, we see that

$$g_i - f_i = d_{i+1} \circ (-P_i) + (-P_{i-1}) \circ d_i.$$

Transitive: If $Q_\bullet$ is a chain homotopy from $g_\bullet$ to a third chain map $h_\bullet$, then $P_\bullet + Q_\bullet$ is a chain homotopy from $f_\bullet$ to $h_\bullet$: that is, given maps $Q_i\colon A_i \to B_{i+1}$ with $g_i - h_i = d_{i+1} \circ Q_i + Q_{i-1} \circ d_i$, we see that

$$f_i - h_i = (f_i - g_i) + (g_i - h_i) = d_{i+1} \circ (P_i + Q_i) + (P_{i-1} + Q_{i-1}) \circ d_i.$$

§2.1 #15. *For an exact sequence $A \to B \to C \to D \to E$ show that $C = 0$ iff the map $A \to B$ is surjective and $D \to E$ is injective.*

Label the maps as

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E.$$

First suppose that $C = 0$. Then $g = 0$, so $\ker g = B$, so $\operatorname{im} f = B$ because the sequence is exact at B, so f is surjective. And $h = 0$, so $\operatorname{im} h = 0$, so $\ker k = 0$ because the sequence is exact at D, so k is injective.

Conversely, suppose that f is surjective and k is injective. Then $\operatorname{im} f = B$, so $\ker g = B$ because the sequence is exact at B, so $g = 0$, so $\operatorname{im} g = 0$. And $\ker k = 0$, so $\operatorname{im} h = 0$ because the sequence is exact at D, so $h = 0$, so $\ker h = C$. But the sequence is exact at C, so $\ker h = \operatorname{im} g$, so $C = 0$.

Hence for a pair of spaces (X, A), the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups iff $H_n(X, A) = 0$ for all n.

The long exact sequence of the pair includes

$$H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X).$$

The maps $H_n(A) \to H_n(X)$ are isomorphisms for all n if and only if they are both injective and surjective for all n. By reindexing, this is true if and only if the leftmost map in our five-term exact sequence is surjective and the rightmost map is injective for all n. But by the first part of the problem, this is true if and only if the middle group vanishes for all n.

§2.1 #20. *Show that $\tilde{H}_n(X) \approx \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X.*

In Chapter 0, Hatcher defined the cone CX as $(X \times I)/(X \times \{0\})$. It is contractible: we see that $X \times I$ deformation retracts onto $X \times \{0\}$ via the homotopy $f_t(x, s) = (x, ts)$, and this descends to a deformation retraction of CX onto the cone point.

He defined the suspension SX as $(X \times I)/(X \times \{0, 1\})$, and we will view it as the quotient of CX by the image of $X \times \{1\}$, which is homeomorphic to X and is a good pair in the sense of Theorem 2.13 or Proposition 2.22: the image of $X \times (0, 1]$ in CX deformation retracts onto the image of $X \times \{1\}$ via a homotopy like the one above.

Thus we get a long exact sequence

$$\cdots \to \tilde{H}_n(CX) \to \tilde{H}_n(SX) \to \tilde{H}_{n-1}(X) \to \tilde{H}_{n-1}(CX) \to \cdots$$

Because CX is contractible, we have $\tilde{H}_n(CX) = 0$ for all n, so the connecting homomorphism $\tilde{H}_n(SX) \to \tilde{H}_{n-1}(X)$ is both injective and surjective for all n, so it is an isomorphism.

More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of n cones CX with their bases identified.

There are a couple of ways to approach this, but here is one.

Let $C_1, \dots, C_n$ be n copies of CX, and let Y be the space obtained by gluing them all together along the images of $X \times \{1\}$ in CX. Then we see that

$$Y/C_1 \cong \overbrace{SX \vee SX \vee \cdots \vee SX}^{n-1 \text{ times}},$$

and the long exact sequence of the pair (Y, C_1) includes

$$\tilde{H}_n(C_1) \to H_n(Y) \to \tilde{H}_n(Y/C_1) \to \tilde{H}_{n-1}(C_1),$$

so

$$\begin{aligned}
\tilde{H}_n(Y) &\cong \tilde{H}_n(Y/C_1) \\
&\cong \tilde{H}_n(SX \vee SX \vee \cdots \vee SX) \\
&\cong \tilde{H}_n(SX) \oplus \tilde{H}_n(SX) \oplus \cdots \oplus \tilde{H}_n(SX) \\
&\cong \tilde{H}_{n-1}(X) \oplus \tilde{H}_{n-1}(X) \oplus \cdots \oplus \tilde{H}_{n-1}(X),
\end{aligned}$$

where the direct sum is taken $n - 1$ times.

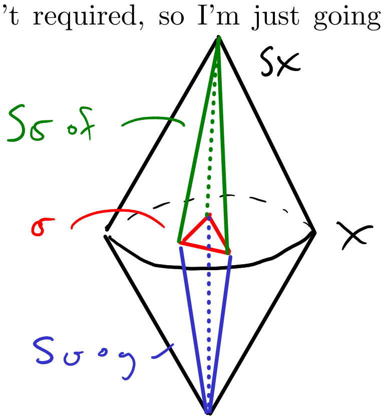
Challenge: §2.1 #21. *Making the preceding problem more concrete, construct explicit chain maps* $s\colon C_n(X) \to C_{n+1}(SX)$ *inducing isomorphisms* $\tilde{H}_n(X) \to \tilde{H}_{n+1}(SX)$.

The idea is illustrated in the picture below: given a triangle in X, we get a tetrahedron above it in SX, and another tetrahedron below it, and we want to take their difference.

Given an n-simplex $\sigma\colon \Delta^n \to X$, we can take the suspension $S\sigma\colon S\Delta^n \to SX$. Divide $S\Delta^n$ into two halves, the images of $\Delta^n \times [0, \frac{1}{2}]$ and $\Delta^n \times [\frac{1}{2}, 1]$, and choose homeomorphisms f from Δ^{n+1} to the first half and g from Δ^{n+1} to the second half. If we choose f and g well, then the map $C_n(X) \to C_{n+1}(X)$ determined by

$$\sigma \mapsto S\sigma \circ f - S\sigma \circ g$$

is a chain map, and induces an isomorphism on homology. But it's late and this problem wasn't required, so I'm just going to assert that the details "can be checked."



Here's another possibility, although maybe it's not quite in the spirit of the problem since it uses the big hammer of Proposition 2.22. Identify $\tilde{H}_{n+1}(SX)$ with $H_{n+1}(CX, X)$, where X is still embedded in CX as the image of $X \times \{1\}$. Given an n-simplex $\sigma\colon \Delta^n \to X$, we can take the cone $C\sigma\colon C\Delta^n \to C_X$. Choose a homeomorphism $h\colon \Delta^{n+1} \to C\Delta^n$; then the map $C_n(X) \to C_{n+1}(CX)$ determined by

$$\sigma \mapsto C\sigma \circ h$$

is *not* a chain map, but if instead we map $C_n(X) \to C_{n+1}(CX, X)$ then it is a chain map. And the induced map $H_n(X) \to H_{n+1}(CX, X)$ turns out to be inverse to the connecting homomorphism $H_{n+1}(CX, X) \to H_n(X)$, hence is an isomorphism.