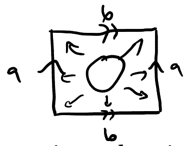
Solutions to Homework 5

§1.2 #9. I'll take $g = 2$ and $h = k = 1$, so the problem reads as follows:

In the surface M_2 of genus 2, let C be a circle that separates M_2 into two compact subsurfaces M_1' and M_1'' homeomorphic to the torus minus an open disk. Show that M_1' does not retract onto C, and hence that M_2 does not retract onto C. But show that M_2 does retract onto the nonseparating circle C' in the figure.

Let $i\colon C \to M_1'$ be the inclusion. We have $\pi_1(C) = \mathbb{Z}$, and on the first homework we saw that M_1' deformation retracts onto $S^1 \vee S^1$, so $\pi_1(M_1')$ is the free group on two letters a and b. Moreover, we see that $i_*\colon \pi_1(C) \to \pi_1(M_1')$ sends $1 \in \mathbb{Z}$ to $aba^{-1}b^{-1}$:

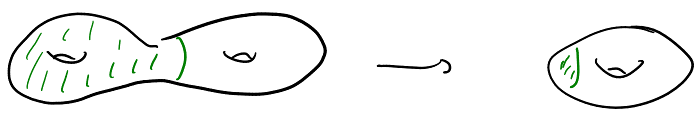


Suppose that $r\colon M_1' \to C$ were a retraction, that is, $r \circ i = 1$. By functoriality, we should have $r_*(i_*(1)) = 1$. But because $r_*\colon \pi_1(M_1') \to \pi_1(C)$ is a group homomorphism and $\pi_1(C) = \mathbb{Z}$, we instead have

$$r_*(aba^{-1}b^{-1}) = r_*(a) + r_*(b) - r_*(a) - r_*(b) = 0.$$

If there were a retraction $\rho\colon M_2 \to C$, then the restriction $\rho|_{M_1'}$ would be a retraction of M_1' onto C.

To retract M_2 onto the non-separating circle C', first map it onto a torus as shown:



Then retract the torus onto C' via

$$\begin{array}{ccc} S^1 \times S^1 & \to & S^1 \\ (z, w) & \mapsto & z. \end{array}$$

§1.3 #8. *Let $\tilde{X}$ and $\tilde{Y}$ be simply-connected coering spaces of the path-connected, locally path-connected spaces X and Y. Show that if $X \simeq Y$ then $\tilde{X} \simeq \tilde{Y}$. [Exercise 11 in Chapter 0 may be helpful.]*

The solution would be cleaner if the homotopy equivalence $X \simeq Y$ were relative to a basepoint, but it's not.

Let $f\colon X \to Y$ and $g\colon Y \to X$ be maps with $g \circ f \simeq 1$ and $g \circ f \simeq 1$. Label the covering maps as $p\colon \tilde{X} \to X$ and $q\colon \tilde{Y} \to Y$. Choose a basepoint $\tilde{x}_0 \in \tilde{X}$, let

$$x_0 = p(\tilde{x}_0), \qquad y_0 = f(y_0), \qquad x_0' = g(x_0), \qquad y_0' = f(x_0'),$$

and choose

$$\tilde{y}_0 \in q^{-1}(y_0), \qquad \tilde{x}_0' \in p^{-1}(x_0'), \qquad \tilde{y}_0' \in q^{-1}(y_0').$$

We want to lift $f \circ p$ through q to get a map $\tilde{f}$ as in the diagram

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[r, dotted, "\tilde{f}"] \arrow[d, "p"'] \arrow[dr, "fp"] & (\tilde{Y}, \tilde{y}_0) \arrow[r, dotted, "\tilde{g}"] \arrow[d, "q"'] \arrow[dr, "qg"] & (\tilde{X}, \tilde{x}_0') \arrow[r, dotted, "\tilde{f}'"] \arrow[d, "p"'] \arrow[dr, "fp"] & (\tilde{Y}, \tilde{y}_0') \arrow[d, "q"'] \\
(X, x_0) \arrow[r, "f"] & (Y, y_0) \arrow[r, "g"] & (X, x_0') \arrow[r, "f"] & (Y, y_0')
\end{tikzcd}$$

Because $\pi_1(\tilde{X}, \tilde{x}_0) = 0$, we have

$$f_* p_* \pi_1(\tilde{X}, \tilde{x}_0) \subset q_* \pi_1(\tilde{Y}, \tilde{y}_0),$$

so the lifting criterion (Proposition 1.33) gives such an $\tilde{f}$. Similarly we get $\tilde{g}$ and $\tilde{f}'$ as in the diagram.

At this point we might like to show that $\tilde{g} \circ \tilde{f} \simeq 1$, but this may not be possible; instead we will show that it is homotopic to a deck transformation, which will be good enough.

We have $g \circ f \simeq 1$, so $g \circ f \circ p \simeq p$; let $F\colon \tilde{X} \times I \to X$ be a homotopy from $g \circ f \circ p$ to p. The map $\tilde{g} \circ \tilde{f}$ lifts $g \circ f \circ p$, so by the homotopy lifting property (Proposition 1.30), there is a unique lift $\tilde{F}\colon \tilde{X} \times I \to \tilde{X}$ of F such that $\tilde{F}(-, 0) = \tilde{g} \circ \tilde{f}$. Let $\varphi = \tilde{F}(-, 1)\colon \tilde{X} \to \tilde{X}$. Then φ lifts $p\colon \tilde{X} \to X$, so it is a deck transformation: for one way to see this, by Proposition 1.37 there is a deck transformation $\tilde{X} \to \tilde{X}$ taking $\tilde{x}_0$ to $\varphi(\tilde{x}_0)$, and by the unique lifting property (Proposition 1.34) this deck transformation and φ must be equal. In particular, φ is a homeomorphism.

Similarly, $\tilde{f}' \circ \tilde{g}$ is homotopic to a deck transformation $\psi\colon \tilde{Y} \to \tilde{Y}$.

From $\tilde{g} \circ \tilde{f} \simeq \varphi$ and $\tilde{f}' \circ \tilde{g} \simeq \psi$ we get

$$\tilde{g} \circ (\tilde{f} \circ \varphi^{-1}) \simeq 1 \qquad \text{and} \qquad (\psi^{-1} \circ \tilde{f}') \circ \tilde{g} \simeq 1,$$

so $\tilde{g}$ is a homotopy equivalence by Chapter 0 #11.

§1.3 #12. *Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^!$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.*

First we observe that the quotient of the free group on a and b by this normal subgroup,

$$\langle a, b \mid a^2 = b^2 = (ab)^4 = 1\rangle$$

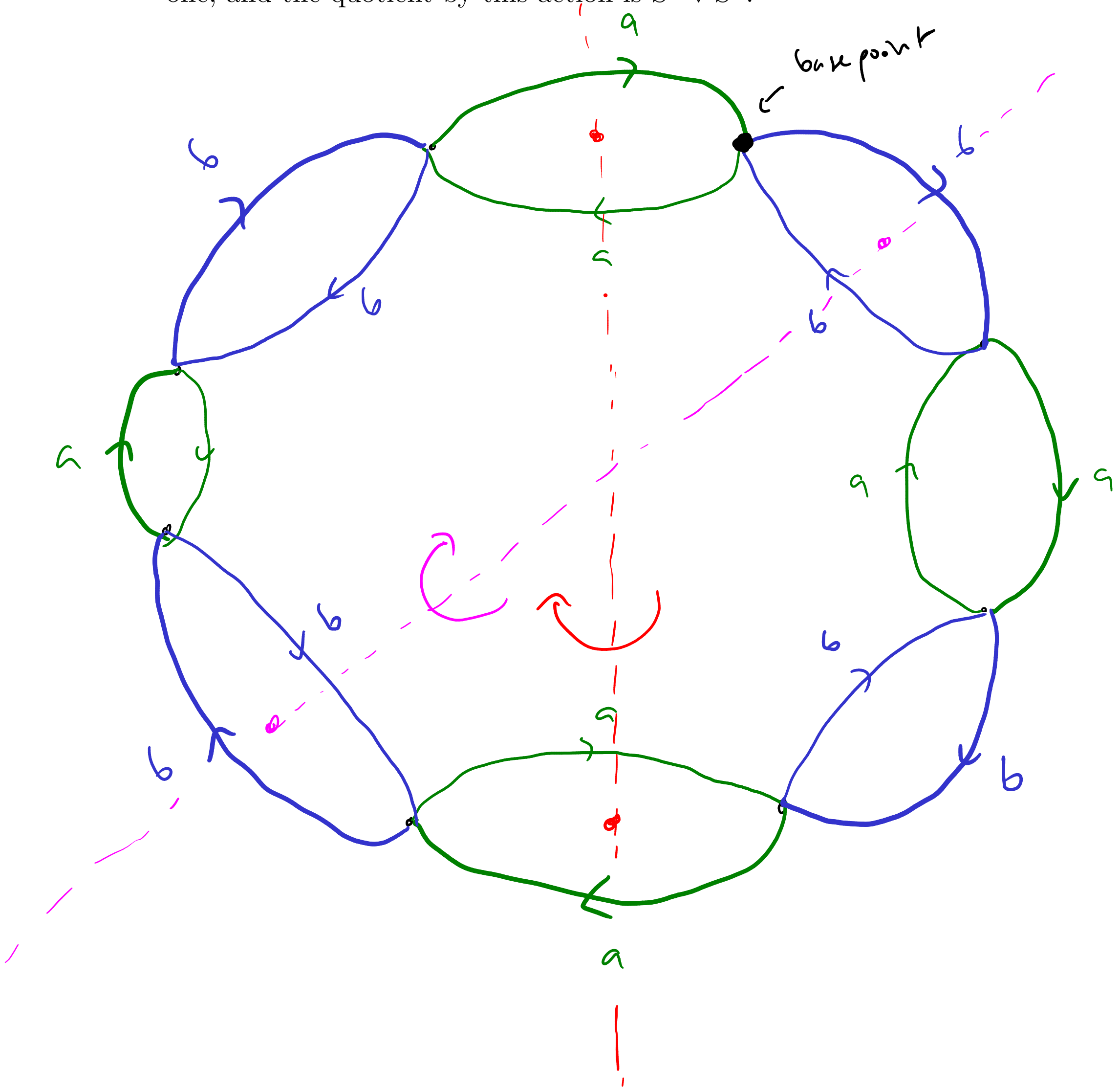
is just the alternative presentation of the dihedral group

$$D_4 = \langle r, s \mid r^4 = s^2 = 1, rs = sr^{-1}\rangle,$$

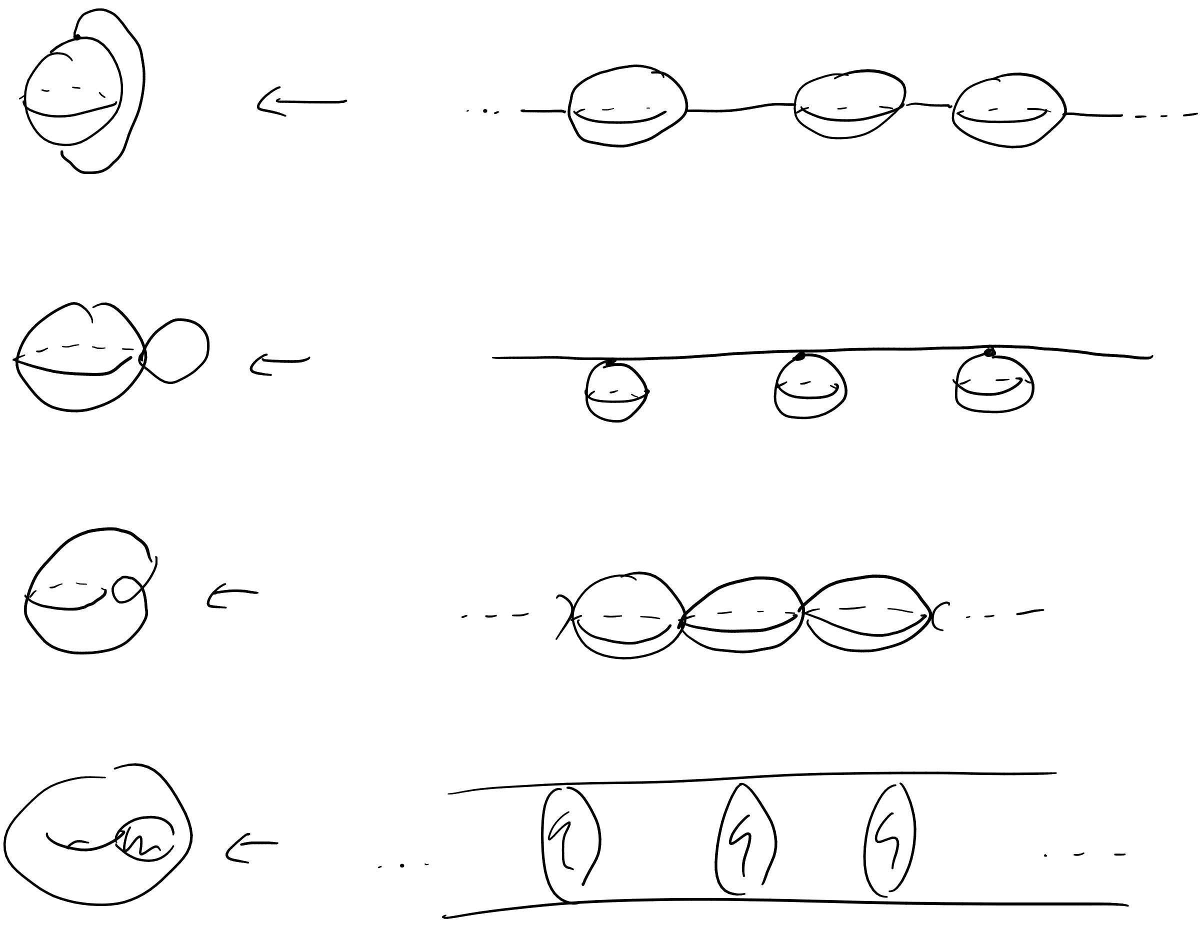
where $a \mapsto s$ and $b \mapsto sr$. So the quotient has order 8.

Our subgroup is the unique normal subgroup of index 8 containing a^2, b^2, and $(ab)^4$, so we seek a normal, 8-sheeted cover $p\colon \tilde{X} \to S^1 \vee S^1$ such that the image of p_* contains those elements; it will necessarily be unique.

Here is such a cover, similar to example (8) in the table on page 58; think of the clockwise edges as arching above the page, and the counterclockwise ones as hanging below it. It is a normal cover because the quotient group acts on it, with a flipping about the red dotted line and b about the pink one, and the quotient by this action is $S^1 \vee S^1$.



Draw a simply-connected cover of each of the four spaces from Homework 3 #4.



(cylinder with discs in it.)