Solutions to Homework 4

§1.1 #8. *Does the Borsuk–Ulam theorem hold for the torus? In other words, for every map $f\colon S^1 \times S^1 \to \mathbb{R}^2$ must there exist $(x, y) \in S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$?*

No, just project onto one factor: let $f\colon S^1 \times S^1 \to \mathbb{R}^2$ be given by $f(x, y) = x$. If $f(x, y) = f(-x, -y)$ then $x = -x$, so $x = 0$, but $0 \notin S^1$.

§1.1 #16. *Show that there are no retractions $r\colon A \to X$ in the following cases.*

A retraction would satisfy $r \circ i = 1$, where i is the inclusion $A \hookrightarrow X$. Thus the induced maps on π_1 would satisfy $r_* \circ i_* = 1$. In particular r_* would be surjective and i_* would be injective.

(a) *$X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.*

We have $\pi_1(A) = \mathbb{Z}$ and $\pi_1(X) = 0$, so i_* cannot be injective.

(b) *$X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.*

We have $\pi_1(A) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(X) = \mathbb{Z}$, so i_* cannot be injective: if $i_*(1, 0) = m \neq 0$ and $i_*(0, 1) = n$ then $i_*(n, -m) = 0$.

(c) *$X = S^1 \times D^2$ and A the circle shown in the figure.*

We have $\pi_1(A) = \mathbb{Z}$ and $\pi_1(X) = \mathbb{Z}$, but the inclusion $i\colon A \to X$ is nullhomotopic, so $i_* = 0$.

(d) *$X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.*

We have $\pi_1(A) = \mathbb{Z} * \mathbb{Z}$ and $\pi_1(X) = 0$, so i_* cannot be injective.

(e) *X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.*

We have $\pi_1(A) = \mathbb{Z} * \mathbb{Z}$, and X is homotopy equivalent to a circle, so $\pi_1(X) = \mathbb{Z}$, so i_* cannot be injective: if a and b are the two free generators of $\pi_1(A)$, and $i_*(a) = m \neq 0$ and $i_*(b) = n$, then $i_*(a^n b^{-m}) = 0$.

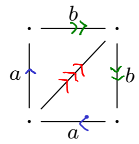
(f) *X the Möbius band and A its boundary circle.*

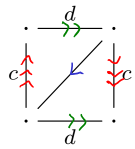
We have $\pi_1(A) = \mathbb{Z}$, and X is homotopy equivalent to a circle, so $\pi_1(X) = \mathbb{Z}$, but the induced map i_* is multiplication by 2. This *is* injective for once, but there is still no map $r_* \colon \mathbb{Z} \to \mathbb{Z}$ with $r_* \circ i_* = 1$.

3. *Show that the groups*

$$\begin{aligned} G &= \langle a, b \mid abba = 1 \rangle \\ H &= \langle c, d \mid cdcd^{-1} = 1 \rangle \end{aligned}$$

are isomorphic. Your proof should be purely algebraic, but it may help to consider the following diagrams:





Looking at the lower right triangle, we let $\varphi \colon G \to H$ be the homomorphism determined by $\varphi(a) = cd^{-1}$ and $\varphi(b) = d$, which is well-defined because

$$\varphi(abba) = cd^{-1} \cdot d \cdot d \cdot cd^{-1} = cdcd^{-1} = 1.$$

Looking at the upper left triangle, we let $\psi \colon H \to G$ be the homomorphism determined by $\psi(c) = ab$ and $\psi(d) = b$, which is well-defined because

$$\psi(cdcd^{-1}) = ab \cdot b \cdot ab \cdot b^{-1} = abba = 1.$$

Then we have

$$\begin{aligned} \psi(\varphi(a)) &= \psi(cd^{-1}) = ab \cdot b^{-1} = a \\ \psi(\varphi(b)) &= \psi(d) = b \end{aligned}$$

$$\begin{aligned} \varphi(\psi(c)) &= \varphi(ab) = cd^{-1} \cdot d = c \\ \varphi(\psi(d)) &= \varphi(b) = d, \end{aligned}$$

so $\psi \circ \varphi = 1_G$ and $\varphi \circ \psi = 1_H$, so φ and ψ are inverse isomorphisms.