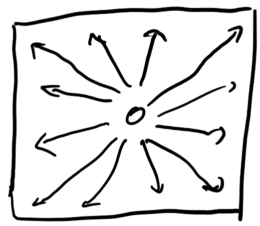
Solutions to Homework 2

Ch. 0 #1. Let X be the solid square $[-1,1] \times [-1,1] \subset \mathbb{R}^2$. First we describe a deformation retraction of $X \setminus 0$ onto its boundary square. Define a map $r\colon X \setminus 0 \to X \setminus 0$ by projecting from origin: explicitly,

$$r(x,y) = \begin{cases} (1,\ y/x) & \text{if } -x \le y \le x \text{ (right quarter)} \\ (x/y,\ 1) & \text{if } -y \le x \le y \text{ (top quarter)} \\ (-1,\ -y/x) & \text{if } x \le y \le -x \text{ (left quarter)} \\ (-x/y, -1) & \text{if } y \le x \le -y \text{ (bottom quarter).} \end{cases}$$



We see that the denominators are non-zero away from the origin, and the four rules agree on the overlaps, hence they patch together give a continuous function on $X \setminus 0$. Moreover we see that r fixes the boundary square, so the straight-line homotopy F from the identity to r is a deformation retraction.

Next show that F descends to a deformation retraction of the punctured torus $S^1 \times S^1 \setminus (-1,-1)$ onto the wedge of circles $S^1 \vee S^1 = (S^1 \times 1) \cup (1 \times S^1)$. The map

$$p\colon X \to S^1 \times S^1 \subset \mathbb{C}^2$$

given by $p(x,y) = (e^{i\pi x}, e^{i\pi y})$ is a quotient map, because the domain is compact and the codomain is Hausdorff. It restricts to give quotient maps from $X \setminus 0$ onto $S^1 \times S^1 \setminus (-1,-1)$ and from ∂X onto $S^1 \vee S^1$. The deformation retraction F descends to give the desired deformation retraction $\bar{F}$ as shown:

$$\begin{tikzcd} (X \setminus 0) \times I \arrow[r, "F"] \arrow[d, "p\times 1"'] & X \setminus 0 \arrow[d, "p"] \\ (S^1 \times S^1 \setminus (-1,-1)) \times I \arrow[r, "\bar{F}"'] & S^1 \times S^1 \setminus (-1,-1). \end{tikzcd}$$

The universal property of the quotient topology applies because $p \circ F$ is constant on the fibers of $p \times 1$.

Ch. 0 #12. First we argue that *any* continuous map $f\colon X \to Y$ induces a map f_* from the set of path components of X to the set of path components of Y, and similarly with connected components. For a point $x \in X$, we let $[x]$ denote the path component or connected component of x, and we define $f_*([x]) = [f(x)]$. This is well-defined, as follows. If x' is another point in the same path component as X, let $\gamma\colon I \to X$ is a path from x to x'; then $f \circ \gamma$ is a path from $f(x)$ to $f(x')$, so these lie in the same path component of Y. If x' is another point in the same connected component as x, let $A \subset X$ be a connected subspace that contains both x and x'; then $f(A)$ is a connected subspace of Y that contains both $f(x)$ and $f(x')$, so these lie in the same connected component of Y.

Next we argue that if $f \simeq f'$ then $f_* = f'_*$ in both cases. Let $F\colon X \times I \to Y$ be a homotopy from f to f'. For path components, we see that for all $x \in X$, the path $F(x, -)$ is a path from $f(x)$ to $f'(x)$, so

$$f_*([x]) = [f(x)] = [f'(x)] = f'_*([x]).$$

For connected components, we see that $F(x \times I)$ is a connected subspace of Y that contains both $f(x)$ and $f'(x)$, so the same equation holds.

Next we observe that if $g\colon Y \to Z$ is another continuous map, then $g_* \circ f_* = (g \circ f)_*$ for either path components or connected components: for any $x \in X$ we have

$$g_*(f_*([x])) = g_*([f(x)]) = [g(f(x))] = (g \circ f)_*([x]).$$

And the identity map $1\colon X \to X$ induces the identity map in both cases:

$$1_*([x]) = [1(x)] = [x].$$

Having set all this up, suppose that $f\colon X \to Y$ is a homotopy equivalence, and let $g\colon Y \to X$ be a homotopy inverse, so $g \circ f \simeq 1$ and $f \circ g \simeq 1$. Then $g_* \circ f_* = (g \circ f)_* = 1_* = 1$, and similarly $f_* \circ g_* = 1$, so f_* and g_* are inverse bijections.

For the claim about when path components and connected components coincide, observe that the diagramp

$$\begin{tikzcd}
\text{path components of } X \arrow[r, "f_*"] \arrow[d] & \text{path components of } Y \arrow[d] \\
\text{connected components of } X \arrow[r, "f_*"] & \text{connected components of } Y
\end{tikzcd}$$

commutes, so if the top, bottom, and left side are bijections, then the right side is a bijection as well.

Ch. 0 #13. Regard S^n as embedded in $\mathbb{R}^{n+1}$ in the usual way, and regard $\mathbb{R}^{m+1} \times \mathbb{R}^{n+1} = \mathbb{R}^{m+n+2}$. Consider the map

$$f\colon S^m \times S^n \times I \to S^{m+n+1}$$

given by

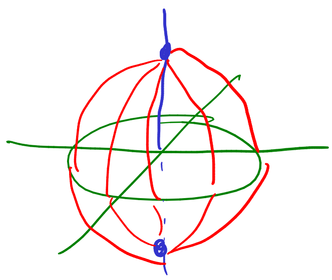
$$f(x, y, t) = (\cos(\tfrac{\pi}{2}t) \cdot x,\ \sin(\tfrac{\pi}{2}t) \cdot y).$$

By definition, the join is the quotient of $S^m \times S^n \times I$ that collapses $S^m \times y \times 0$ for each $y \in S^n$, and $x \times S^n \times 1$ for each $x \in S^m$. We will see that f is surjective, and the fibers are exactly these equivalence classes. Because it is a continuous map from a compact space onto a Hausdorff space, it identifies the quotient space of $S^m \times S^n \times I$ with S^{m+n+1}.

So we analyze the fibers of f. Let $(v, w) \in S^{m+n+1}$, so $v \in \mathbb{R}^{m+1}$, $w \in \mathbb{R}^{n+1}$, and $|v|^2 + |w|^2 = 1$. If $v \neq 0$ and $w \neq 0$, let θ be the unique angle between 0 and $\frac{\pi}{2}$ with $\cos\theta = |v|$ and $\sin\theta = |w|$; then

$$f\left(\frac{v}{|v|}, \frac{w}{|w|}, \frac{\theta}{\pi/2}\right) = (v, w),$$

so $f^{-1}(v, w)$ is a single point. If $v = 0$ then $w \in S^n$, and we see that $f^{-1}(0, w)$ is $S^m \times w \times 1$. If $w = 0$ then $v \in S^m$, and we see that $f^{-1}(v, 0)$ is $v \times S^n \times 0$.



$S^1 * S^0 = S^2$

§1.1 #5 We can deduce (a) $\Leftrightarrow$ (b) from problem 2 from last week. Consider the diagram

$$\begin{tikzcd} S^1 \times I \arrow[d, "f"'] \arrow[dr, dotted, "F"] & \\ D^2 \arrow[r, "h"'] & X \end{tikzcd}$$

where $f(z,t) = tz$ collapses $S^1 \times 0$ to a point. Let $i\colon S^1 \hookrightarrow D^1$ be the inclusion of the boundary circle, and let

$$j\colon S^1 \times 1 \hookrightarrow S^1 \times I;$$

Then $f \circ j = i$. Suppose we are given a map $g\colon S^1 \to X$. By definition, g is nullhomotopic if and only if there is a map $F\colon S^1 \times I \to X$ such that $F(x,0)$ is constant and $F \circ j = g$. By the universal property of the quotient map f, this happens if and only if there is a map $h\colon D^2 \to X$ such that $h \circ f = F$, and $h \circ f \circ j = g$. But $f \circ j = i$, so by definition h is a map that extends g in the sense of (b).

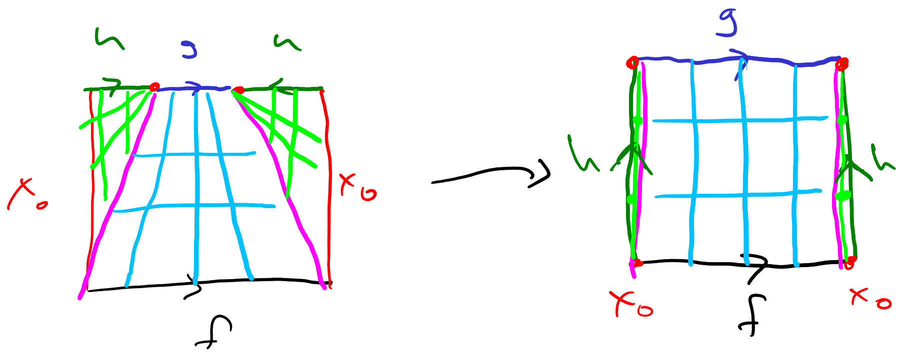
We can deduce (a) $\Leftrightarrow$ (c) from problem 6 below. For a point $x_0 \in X$, let $P \subset X$ be the path component of x_0, so $\pi_1(X, x_0) = \pi_1(P, x_0)$. Problem 6 implies that (c) holds if and only if $[S^1, P]$ is a one-point set for every path component $P \subset X$; note that a group is trivial if and only if all its elements are conjugate. And $[S^1, P]$ is a one-point set if and only if (a) holds; note that any two constant maps $S^1 \to P$ are homotopic because P is path connected.

Finally we argue that X is simply-connected if and only if all maps $S^1 \to X$ are homotopic. By definition, simply-connected means "path-connected and (c)," which is equivalent to "path-connected and (a)." We see that X is path-connected if and only if all constant maps to X are homotopic, so the latter is equivalent to "all maps $S^1 \to X$ are homotopic."

§1.1 #6 We will pass back and forth freely between maps $\ell\colon S^1 \to X$ and maps $\gamma\colon I \to X$ with $\gamma(0) = \gamma(1)$, as on Worksheet 5: the correspondence is $\gamma = \ell \circ p$, where $p\colon I \to S^1$ is the quotient map given by $p(s) = e^{2\pi i s}$. Similarly we pass between homotopies $\ell_0 \simeq \ell_1$ and homotopies $\gamma_0 \simeq \gamma_1$ that satisfy $F(0,t) = F(1,t)$ for all $t \in I$, using the quotient map $(p \times 1)\colon I \times I \to S^1 \times I$.

First we show that if X is path-connected then Φ is onto. Suppose that $\ell\colon S^1 \to X$ corresponds to a path $\gamma\colon I \to X$ with $\gamma(0) = \gamma(1)$. Choose a path $p\colon I \to X$ with $p(0) = x_0$ and $p(1) = \gamma(0)$. I claim that Φ sends $[p \cdot \gamma \cdot \bar{p}] \in \pi_1(X, x_0)$ to $[\ell] \in [S^1, X]$, where $\bar{p}$ is the inverse path.* To see this, observe that the paths $p \cdot \gamma \cdot \bar{p}$ and $\gamma \cdot p \cdot \bar{p}$ give homotopic maps $S^1 \to X$: just rotate the circle by 120°. Now $\gamma \cdot p \cdot \bar{p}$ is homotopic rel. endpoints to γ, as we saw in the construction of π_1.

Next we show that if $f, g\colon I \to X$ are loops based at x_0 and $\Phi([f]) = \Phi([g])$, then $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Let $F\colon I \times I \to X$ be a homotopy between f and g satisfying $F(0,t) = F(1,t)$ for all $t \in I$. Let $h\colon I \to X$ be the path $h(t) = F(0,t)$. Then $h(0) = h(1) = x_0$. I claim that F gives a homotopy rel. endpoints from f to $h \cdot g \cdot \bar{h}$, so $[f] = [h][g][\bar{h}] \in \pi_1(X, x_0)$. To prove the claim, pre-compose F with the map $I \times I \to I \times I$ indicated below:



More precisely but less clearly, this is

$$(s,t) \mapsto \begin{cases} (0,\ 3s) & \text{if } t \geq 3s, \\ (1,\ 3-3s) & \text{if } t \geq 3-3s, \\ \left(\frac{s-\frac{1}{3}t}{1-\frac{2}{3}t},\ t\right) & \text{otherwise.} \end{cases}$$

*In lecture I called this p^{-1}, but here I decided to match Hatcher's notation.