Solutions to Homework 1

1. (a) Let X and Y be topological spaces. Define what it means for a map $f\colon X \to Y$ to be a *quotient map*.

   Solution: f is surjective, and for every subset $U \subset Y$, U is open if and only if $f^{-1}(U)$ is open.

   (b) Let $f\colon X \to Y$ be a quotient map. Show that for any other space Z, a continuous map $g\colon X \to Z$ descends to Y – that is, there is a continuous map $h\colon Y \to Z$ with $g = h \circ f$ – if and only if g is constant on the fibers of f.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "h"'] & Z \end{tikzcd}$$

   Solution: First suppose that $g = h \circ f$. If $x, x' \in X$ lie in the same fiber of f, then $g(x) = h(f(x)) = h(f(x')) = g(x')$.

   Conversely, suppose that g is constant on the fibers of f. Given a point $y \in Y$, choose a point $x \in X$ with $f(x) = y$, which is possible because f is surjective, and define $h(y) = g(x)$. Then h is well-defined: if $x' \in X$ is another point with $f(x') = y$, then $g(x') = g(x)$ by hypothesis.

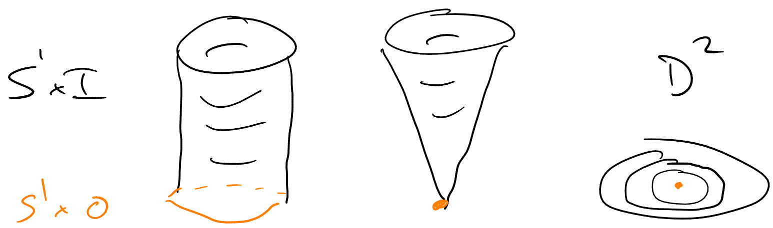
   It remains to show that h is continuous. Let $U \subset Z$ be open. Then $f^{-1}(h^{-1}(U)) = g^{-1}(U)$ is open because g is continuous, so $h^{-1}(U)$ is open because f is a quotient map.

   (c) Suppose that X is compact, Y is Hausdorff, and $f\colon X \to Y$ is a continuous surjection. Show that f is a quotient map.

   Solution: We will show that $F \subset Y$ is closed if and only if $f^{-1}(F)$ is closed, which is equivalent to the definition because f^{-1} preserves complements. If $F \subset Y$ is closed then $f^{-1}(F)$ is closed because f is continuous. Conversely, suppose that $f^{-1}(F)$ is closed. A closed subset of a compact space is compact, so $f^{-1}(F)$ is compact. Because f is surjective, we have $F = f(f^{-1}(F))$, and the continuous image of a compact space is compact, so F is compact. A compact subspace of a Hausdorff space is closed, so F is closed.

2. I claim that $(S^1 \times I)/(S^1 \times \{0\})$ is homeomorphic to D^2.

   (a) Draw a picture.
   Solution:

   (b) Argue that it is enough to write down a continuous surjection $S^1 \times I \to D^2$ which is constant on $S^1 \times \{0\}$ and injective otherwise.
   Solution: Let $f\colon S^1 \times I \to D^2$ be such a map, and let

   $$g\colon S^1 \times I \to (S^1 \times I)/(S^1 \times \{0\})$$

   be the natural map. The fibers of f and g are the same: $S^1 \times \{0\}$ and the single points of $S^1 \times (0, 1]$; thus f is constant on the fibers of g and vice versa. By problem 1(c), f is a quotient map, so by problem 1(b) there is a continuous map

   $$h\colon D^2 \to (S^1 \times I)/(S^1 \times \{0\})$$

   with $g = h \circ f$. And g is a quotient map by definition, so there is a continuous map

   $$k\colon (S^1 \times I)/(S^1 \times \{0\}) \to D^2$$

   with $f = k \circ g$. Now we have $f = k \circ h \circ f$ and $g = h \circ k \circ g$, and because f and g are surjective we get $k \circ h = 1$ and $h \circ k = 1$. so h and k are inverse homeomorphisms.

(c) Write down such a map.

Solution: We regard S^1 and D^2 as subsets of $\mathbb{R}^2$ in the usual way. The map we want is

$$\begin{aligned} S^1 \times I &\to D^2 \\ (x,t) &\mapsto tx. \end{aligned}$$

3. (a) Show that any group homomorphism $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ is of the form $\varphi(x) = nx$ for some $n \in \mathbb{Z}$.

Solution: Let $n = \varphi(1)$. For any $x \in \mathbb{Z}$, write

$$x = \overbrace{1 + \cdots + 1}^{x \text{ times}}.$$

Then

$$\varphi(x) = \overbrace{n + \cdots + n}^{x \text{ times}} = nx.$$

(b) Let $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ be multiplication by 2. Show that there is no homomorphism $\psi\colon \mathbb{Z} \to \mathbb{Z}$ such that $\psi \circ \varphi = 1$.

Solution: If there were such a ψ then φ would be surjective: for every $x \in \mathbb{Z}$ we would have $x = \varphi(\psi(x))$. But φ is not surjective: for example, 3 is not in the image of φ.

(c) Let G be a finite group. Show that any homomorphism $\varphi\colon G \to \mathbb{Z}$ is trivial.

Solution: Let $g \in G$. Because G is finite, there is a positive integer n such that $g^n = 1$. Then

$$0 = \varphi(g^n) = n\varphi(g),$$

so $\varphi(g) = 0$.