Homework 1

Due Monday, October 5, 2020

This is just a little review. Future homeworks will be more substantial.

1. (a) Let X and Y be topological spaces. Define what it means for a map $f\colon X \to Y$ to be a *quotient map*.

   (b) Let $f\colon X \to Y$ be a quotient map. Show that for any other space Z, a continuous map $g\colon X \to Z$ descends to Y – that is, there is a continuous map $h\colon Y \to Z$ with $g = h \circ f$ – if and only if g is constant on the fibers of f.

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "h"'] & Z \end{tikzcd}$$

   (c) Suppose that X is compact, Y is Hausdorff, and $f\colon X \to Y$ is a continuous surjection. Show that f is a quotient map.

2. I claim that $(S^1 \times I)/(S^1 \times \{0\})$ is homeomorphic to D^2.

   (a) Draw a picture.

   (b) Argue that it is enough to write down a continuous surjection $S^1 \times I \to D^2$ which is constant on $S^1 \times \{0\}$ and injective otherwise.

   (c) Write down such a map.

3. (a) Show that any group homomorphism $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ is of the form $\varphi(x) = nx$ for some $n \in \mathbb{Z}$.

   (b) Let $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ be multiplication by 2. Show that there is no homomorphism $\psi\colon \mathbb{Z} \to \mathbb{Z}$ such that $\psi \circ \varphi = 1$.

   (c) Let G be a finite group. Show that any homomorphism $\varphi\colon G \to \mathbb{Z}$ is trivial.