Solutions to Final Exam

1. Some homological algebra.

   (a) The *cokernel* of a homomorphism $f\colon A \to B$ is defined to be $B/\operatorname{im} f$. Show that the quotient map $q\colon B \to \operatorname{coker} f$ enjoys the following universal property: we have $q \circ f = 0$, and for every map $\varphi\colon B \to Y$ with $\varphi \circ f = 0$, there is a unique map $\psi\colon \operatorname{coker} f \to Y$ such that $\psi \circ q = \varphi$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "q"] \arrow[dr, "\varphi"'] & \operatorname{coker} f \arrow[d, dotted, "\psi"] \\ & & Y \end{tikzcd}$$

Solution: First we show that $q \circ f = 0$: if $a \in A$ then $f(a) \in \operatorname{im} f$, so $q(f(a)) = 0$.

Next, let $\varphi\colon B \to Y$ be a homomorphism with $\varphi \circ f = 0$, and us construct $\psi\colon \operatorname{coker} f \to Y$. Every element of $\operatorname{coker} f$ is of the form $q(b)$ for some $b \in B$, and we define $\psi(q(b)) = \varphi(b)$. This is well-defined, for if $q(b) = q(b')$ then $q(b - b') = 0$, so $b - b' \in \operatorname{im} f$, so we can write $b - b' = f(a)$ for some $a \in A$, so $\varphi(b) - \varphi(b') = \varphi(f(a)) = 0$, so $\varphi(b) = \varphi(b')$. By construction we have $\psi \circ q = \varphi$.

For uniqueness, let $\psi'\colon \operatorname{coker} f \to Y$ be another map with $\psi' \circ q = \varphi$. Because q is surjective, we find that $\psi' = \psi$.

If you want, you can verify that ψ is a homomorphism: let $q(b_1)$ and $q(b_2)$ be two elements of $\operatorname{coker} f$; then

$$\begin{aligned} \psi(q(b_1) + q(b_2)) &= \psi(q(b_1 + b_2)) = \varphi(b_1 + b_2) \\ &= \varphi(b_1) + \varphi(b_2) = \psi(q(b_1)) + \psi(q(b_2)). \end{aligned}$$

And if we're dealing with modules over some ring R, let $r \in R$ and $q(b) \in \operatorname{coker} f$; then

$$\psi(r \cdot q(b)) = \psi(q(rb)) = \varphi(rb) = r \cdot \varphi(b) = r \cdot \psi(q(b)).$$

(b) State, but do not prove, the dual universal property for the inclusion $i\colon \ker f \to A$.

Solution: We have $f \circ i = 0$, and for every map $\varphi\colon X \to A$ with $f \circ \varphi = 0$, there is a unique map $\psi\colon X \to \ker f$ such that $i \circ \psi = \varphi$.

$$\begin{tikzcd}
X \arrow[d, dotted, "\psi"'] \arrow[dr, "\varphi"] & & \\
\ker f \arrow[r, "i"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

(c) Show that if

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

is an exact sequence, then $\ker h = \operatorname{im} g \cong \operatorname{coker} f$.

Solution: Because the sequence is exact at C, we have $\ker h = \operatorname{im} g$. By the first isomorphism theorem, $\operatorname{im} g \cong B/\ker g$. Because the sequence is exact at B, we have $\ker g = \operatorname{im} f$, so $\operatorname{im} g \cong B/\operatorname{im} f = \operatorname{coker} f$.

(d) Show that if

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \to 0$$

is an exact sequence, then $A \cong \ker g$ and $D \cong \operatorname{coker} g$.

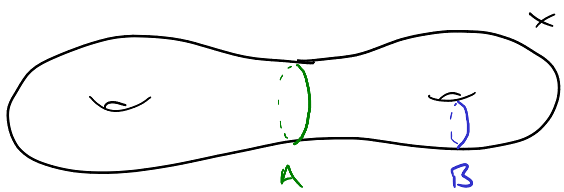
Solution: Because the sequence is exact at A, we have

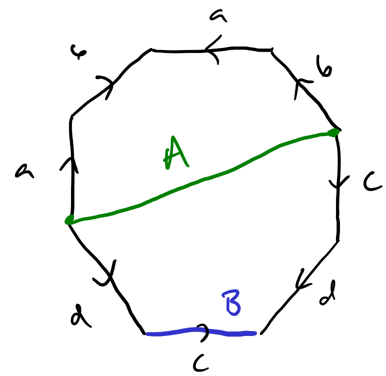
$$\ker f = \operatorname{im}(0 \to A) = 0,$$

so f is injective, so $A \cong \operatorname{im} f$. Because the sequence is exact at B, we have $\operatorname{im} f = \ker g$. Because the sequence is exact at C and D, part (c) gives

$$\operatorname{coker} g \cong \ker(D \to 0) = D.$$

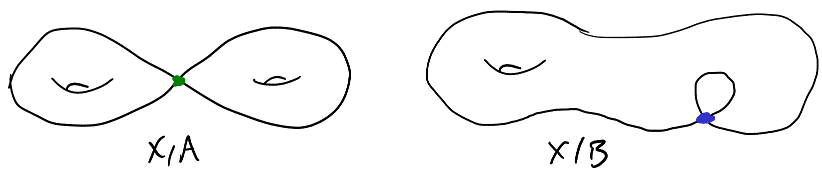
2. Let X be the closed orientable surface of genus 2, and let $i\colon A \hookrightarrow X$ and $j\colon B \hookrightarrow X$ be the inclusion of the separating and non-separating circles indicated below:





(a) Draw pictures of X/A and of X/B.

Solution:



(b) Describe $i_*\colon H_1(A) \to H_1(X)$ and $j_*\colon H_1(B) \to H_1(X)$, and find their kernels and cokernels.

Hint: In Lecture 27, I argued that the boundary of the octagon above gives an inclusion $S^1 \vee S^1 \vee S^1 \vee S^1 \hookrightarrow X$ that induces an isomorphism on H_1, and in particular $H_1(X) \cong \mathbb{Z}^4$.

Solution: Let $a, b, c, d \in C_1(X)$ be as in the octagonal diagram, so $[a], [b], [c], [d]$ are a basis for $H_1(X) \cong \mathbb{Z}^4$ by what we said in Lecture 27. Then j_* takes the generator of $H_1(B) \cong \mathbb{Z}$ to $[c]$. Thus j_* is injective, so $\ker j_* = 0$, and

$$\operatorname{im} j_* = 0 \oplus 0 \oplus \mathbb{Z} \oplus 0 \subset \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

so $\operatorname{coker} j_* = \mathbb{Z}^3$.

Next, I claim that i_* takes a generator of $H_1(A) \cong \mathbb{Z}$ to $[a] + [b] - [a] - [b] = 0$. There are several ways to explain this, and I won't be very strict about how you do it. One way is to say that the upper half of the octagon gives a chain in X whose boundary is $a+b-a-b \pm i_\sharp \alpha$, where $\alpha \in C_1(A)$ is a chain whose homology class $[\alpha]$ generates $H_1(A)$. Thus $i_\sharp \alpha \in C_1(X)$ is a boundary, so $i_*[\alpha] \in H_1(X)$ is zero.

In any event, $i_*\colon \mathbb{Z} \to \mathbb{Z}^4$ is the zero map, so $\ker i_* = \mathbb{Z}$ and $\operatorname{coker} i_* = \mathbb{Z}^4$.

(c) Use the long exact sequence of the pair (X, B) to compute $\tilde{H}_*(X/B)$.

Solution: The long exact sequence in reduced homology looks like this:

$$
\begin{tikzcd}
0 \arrow[r] & \tilde{H}_2(B) \arrow[r, "j_*"] & \tilde{H}_2(X) \arrow[r] & H_2(X,B) \arrow[dll, out=-10, in=170] \\
& \tilde{H}_1(B) \arrow[r, "j_*"] & \tilde{H}_1(X) \arrow[r] & H_1(X,B) \arrow[dll, out=-10, in=170] \\
& \tilde{H}_0(B) \arrow[r, "j_*"] & \tilde{H}_0(X) \arrow[r] & H_0(X,B) \arrow[r] & 0.
\end{tikzcd}
$$

Plugging in what we know about $\tilde{H}_*(B)$ and $\tilde{H}_*(X)$, and using the isomorphism $H_*(X, B) \cong \tilde{H}_*(X/B)$, this becomes

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \tilde{H}_2(X/B) \arrow[dll, out=-10, in=170] \\
& \mathbb{Z} \arrow[r, "j_*"] & \mathbb{Z}^4 \arrow[r] & \tilde{H}_1(X/B) \arrow[dll, out=-10, in=170] \\
& 0 \arrow[r] & 0 \arrow[r] & \tilde{H}_0(X/B) \arrow[r] & 0,
\end{tikzcd}
$$

First we observe that $\tilde{H}_0(X/B) = 0$. Next, using problem 1(d) we find that

$$\tilde{H}_1(X/B) = \operatorname{coker} j_* = \mathbb{Z}^3.$$

Finally, because j_* is injective, the connecting homomorphism $\tilde{H}_2(X/B) \to \mathbb{Z}$ is zero, so the map $\mathbb{Z} \to \tilde{H}_2(X/B)$ is an isomorphism. Thus

$$\tilde{H}_n(X/B) = \begin{cases} 0 & n = 0 \\ \mathbb{Z}^3 & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

This is consistent with the fact that $X/B \simeq (\text{torus}) \vee S^1$, like what saw in Homework 3 #4. It is also interesting to notice that $X/B \cong (\text{torus})/(\text{two points})$, and to get the same answer using the long exact sequence of that pair, like what we did on Worksheet 18.

(d) Do the same with X/A.

Solution: Now the long exact sequence becomes

$$
\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \tilde{H}_2(X/A) \arrow[dll] \\
 & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}^4 \arrow[r] & \tilde{H}_1(X/A) \arrow[dll] \\
 & 0 \arrow[r] & 0 \arrow[r] & \tilde{H}_0(X/A) \arrow[r] & 0,
\end{tikzcd}
$$

First we observe that $\tilde{H}_0(X/A) = 0$. Next, because $i_*\colon \mathbb{Z} \to \mathbb{Z}^4$ is zero, the map $\mathbb{Z}^4 \to \tilde{H}_1(X/A)$ is an isomorphism, and we have a short exact sequence

$$0 \to \mathbb{Z} \to \tilde{H}_2(X/A) \to \mathbb{Z} \to 0.$$

This is necessarily split, as Hatcher explains on pages 147–148, so $\tilde{H}_2(X/A) = \mathbb{Z}^2$. Thus

$$\tilde{H}_n(X/A) = \begin{cases} 0 & n = 0 \\ \mathbb{Z}^4 & n = 1 \\ \mathbb{Z}^2 & n = 2 \\ 0 & \text{otherwise,} \end{cases}$$

which is consistent with the fact that $X/A \cong (\text{torus}) \vee (\text{torus})$.