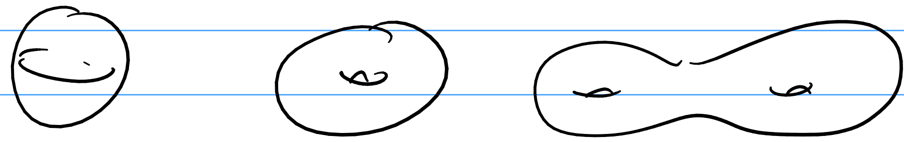
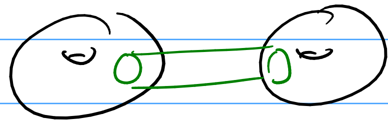
Last time:

for oriented surfaces of genus g



$g=0$ $\quad$ $g=1$ $\quad$ $g=2$

$$\begin{array}{llll} H_* = & \mathbb{Z} & \mathbb{Z}^{2g} & \mathbb{Z} \\ \text{or} & \mathbb{Q} & \mathbb{Q}^{2g} & \mathbb{Q} \\ \text{or} & \mathbb{Z}_2 & \mathbb{Z}_2^{2g} & \mathbb{Z}_2 \end{array}$$



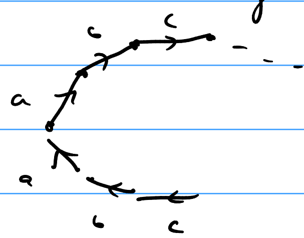
$(\text{torus})^{\# g}$ ...

also draw a $4g$-gon with $\xrightarrow[a]{}\ \xrightarrow[b]{}\ \xleftarrow[a]{}\ \xleftarrow[b]{}$ around the edge

non-orientable surfaces

$\mathbb{RP}^2$ $\quad$ Klein bottle $= \mathbb{RP}^2 \# \mathbb{RP}^2$ $\quad$ $(\mathbb{RP}^2)^{\# n}$

or draw a $2n$-gon with



$$\begin{array}{llll} H_* = & \mathbb{Z}, & \mathbb{Z}_2 \oplus \mathbb{Z}^{n-1} & 0 \\ \text{or} & \mathbb{Q}, & \mathbb{Q}^{n-1} & 0 \\ \text{or} & \mathbb{Z}_2, & \mathbb{Z}_2^{n} & \mathbb{Z}_2 \end{array}$$

on worksheet

$\mathbb{Z} \xrightarrow{\begin{pmatrix}2\\2\\2\end{pmatrix}} \mathbb{Z}^3$

is injective.

cokernel ... think a bit.

OTOH $\mathbb{Q} \xrightarrow{\begin{pmatrix}2\\2\\2\end{pmatrix}} \mathbb{Q}^3$

is a rank-1 map of v.s.

ker $= 0$ $\quad$ coker $= \mathbb{Q}^2$

$\mathbb{Z}_2 \xrightarrow{\begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}} \mathbb{Z}_2^3$

ker $= \mathbb{Z}_2$ $\quad$ coker $= \mathbb{Z}_2^3$

torus # $\mathbb{RP}^2$ = $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$ (fun exercise)

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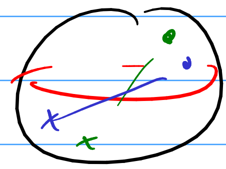
homology of $\mathbb{RP}^n$

$\mathbb{RP}^n = S^n/\pm 1$



= disc glued to $S^{n-1}/\pm 1 = \mathbb{RP}^{n-1}$

homology inductively

$\mathbb{RP}^2$ =

$\tilde H_*(\mathbb{RP}^1) \longrightarrow \tilde H_*(\mathbb{RP}^2) \longrightarrow \tilde H_*(\mathbb{RP}^2/\mathbb{RP}^1)$, $\quad \mathbb{RP}^2/\mathbb{RP}^1 = S^2$

	$\tilde H_*(\mathbb{RP}^1)$	$\tilde H_*(\mathbb{RP}^2)$	$\tilde H_*(\mathbb{RP}^2/\mathbb{RP}^1)$	
2	0 0 0	0 0 $\mathbb{Z}_2$	$\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{Z}_2$	$e \mapsto 2a$
1	$\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{Z}_2$	$\mathbb{Z}_2$ 0 $\mathbb{Z}_2$	0 0 0	
0	0 0 0	0 0 0	0 0 0	

(0) $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ ($\mathbb{Z}_2$) $\qquad$ (0) $\mathbb{Q} \xrightarrow[\cong]{2} \mathbb{Q}$ (0)

($\mathbb{Z}_2$) $\mathbb{Z}_2 \xrightarrow{2=0} \mathbb{Z}_2$ ($\mathbb{Z}_2$)

$\tilde H_*(\mathbb{RP}^2) \longrightarrow \tilde H_*(\mathbb{RP}^3) \longrightarrow \tilde H_*(\mathbb{RP}^3/\mathbb{RP}^2)$, $\quad \mathbb{RP}^3/\mathbb{RP}^2 = S^3$

	$\tilde H_*(\mathbb{RP}^2)$	$\tilde H_*(\mathbb{RP}^3)$	$\tilde H_*(\mathbb{RP}^3/\mathbb{RP}^2)$	
3	0 0 0	$\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{Z}_2$ $\xrightarrow{\cong}$	$\mathbb{Z}$ $\mathbb{Q}$ $\mathbb{Z}_2$	0 ?
2	0 0 $\mathbb{Z}_2$	0 0 $\mathbb{Z}_2$	0 0 0	1 ?
1	$\mathbb{Z}_2$ 0 $\mathbb{Z}_2$ $\xrightarrow{\cong}$	$\mathbb{Z}_2$ 0 $\mathbb{Z}_2$	0 0 0	2 = 0 ...
0	0 0 0	0 0 0	0 0 0	

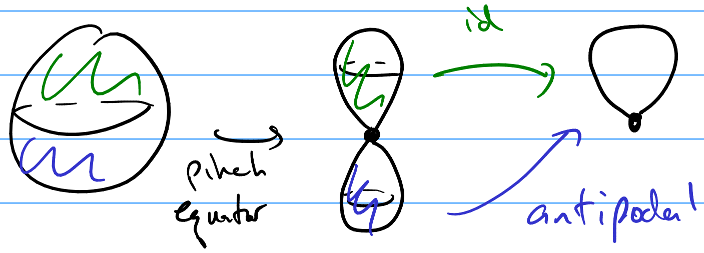
=

$$\tilde H_*(\mathbb{RP}^3) \longrightarrow \tilde H_*(\mathbb{RP}^4) \longrightarrow H_*(\mathbb{RP}^4/\mathbb{RP}^3) \qquad (\cong S^4)$$

	$\tilde H_*(\mathbb{RP}^3)$	$\tilde H_*(\mathbb{RP}^4)$	$H_*(\mathbb{RP}^4/\mathbb{RP}^3)$
4	0	0	$\mathbb{Z}$
3	$\mathbb{Z}$	$\mathbb{Z}_2$	0
2	0	0	0
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0
0	0	0	0

$2 \cdots$

$$\partial D^4 = S^3 \xrightarrow{\text{attach}} \mathbb{RP}^3 \xrightarrow{\text{quot}} \mathbb{RP}^3/\mathbb{RP}^2 = S^3$$



antipodal map induces $+1$ on $H_3(S^3)$

$1+1=2$

in general, find that

$H_*(\mathbb{RP}^{\text{even}}) = \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \dots \mathbb{Z}_2, 0$

$\mathbb{Q}, 0, 0, 0, 0, \dots 0, 0$

$\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \quad \mathbb{Z}_2, \mathbb{Z}_2$

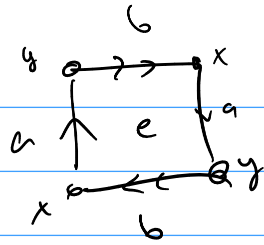
$H_*(\mathbb{RP}^{\text{odd}}) = \mathbb{Z}, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, \dots, \mathbb{Z}_2, 0, \mathbb{Z}$

$\mathbb{Q}, 0, 0, 0, 0, \dots, 0, 0, \mathbb{Q}$

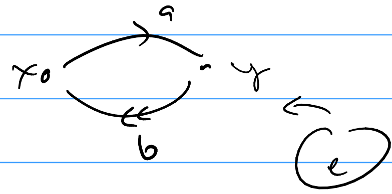
$\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \dots, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2$

better bookkeeping: cellular homology.

Celluler Homology



$\mathbb{RP}^2$



X = CW complex

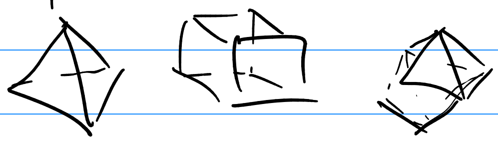
$$X_0 \subset X_1 \subset X_2 \subset \cdots$$

points   1-skeleton   2-skeleton

$$X_i / X_{i-1} = S^i \vee \cdots \vee S^i \quad (\#\text{ of } i\text{-cells})$$

Euler characteristic $\chi(X) = \sum (-1)^i (\#\ i\text{-cells})$

for a polyhedron: # vertices − # edges + # faces

 etc.

eventually: $\chi = \sum (-1)^i \dim H_i(X, F)$ F any field.

So inv. of the space, not cell structure.

Thm (Celluler Homology)

∃ a chain complex to compute $H_i(X; A)$ coeffs, not rel hom.

$$\cdots \longrightarrow A^{\#3\text{ cells}} \xrightarrow{d} A^{\#2\text{ cells}} \xrightarrow{d} A^{\#1\text{-cells}} \xrightarrow{d} A^{\#0\text{ cells}} \longrightarrow 0$$

d comes from attaching maps.

$d^2 = 0$ and $\ker d_i / \operatorname{im} d_{i+1} = H_i(X;A)$

in particular, if X is a finite CW complex then $H_*(X)$ is f.n. gen.

Proof: consider the diagram

$$\begin{tikzcd}
 & & & & & H_i(X^{i+1},X^i)=0 & & \\
 & & H_i(X^{i-1})=0 \arrow[dr] & & H_i(X^{i+1}) = H_i(X) = \operatorname{coker}\partial_{i+1} \arrow[ur] & & & \\
 & & & H_i(X^i) \arrow[ur] \arrow[drr, "q_i \text{ injective}"] & & & & \\
\cdots \arrow[r] & H_{i+1}(X^{i+1},X^i) = \mathbb{Z}^{\#\, i+1 \text{ cells}} \arrow[urr, "\partial_{i+1}"] \arrow[rrrr, "d_{i+1}"'] & & & & H_i(X^i,X^{i-1}) = \mathbb{Z}^{\#\, i\text{-cells}} \arrow[rr, "d_i"] \arrow[dr, "\partial_i"] & & H_{i-1}(X^{i-1},X^{i-2}) = \mathbb{Z}^{\#\, i-1 \text{ cells}} \\
 & & & & & & H_{i-1}(X^{i-1}) \arrow[ur, "q_{i-1}"] & \\
 & & & & & H_{i-1}(X_{i-2})=0 \arrow[ur] & &
\end{tikzcd}$$

$$d_{i+1} := q_i \circ \partial_{i+1}$$

$$d_i \circ d_{i+1} = q_{i-1} \circ \cancel{\partial_i \circ q_i} \circ \partial_{i+1} = 0$$

$$\frac{\ker d_i}{\operatorname{im} d_{i+1}} \overset{\text{want}}{=} H_i(X)$$

$$H_i(X) = \operatorname{coker}\partial_{i+1} = H_i(X^i)/\operatorname{im}\partial_{i+1}$$

$\ker d_i$? $\quad \operatorname{im} q_i = \ker \partial_i$

since q_{i-1} is injective, $\ker d_i = \ker \partial_i =$

$= \operatorname{im} q_i = H_i(X^i)$

$$\operatorname{im}\partial_{i+1} \longleftrightarrow q_i(\operatorname{im}\partial_{i+1}) = \operatorname{im}(q_i \partial_{i+1}) = \operatorname{im} d_{i+1}$$

$$\ker d_i / \operatorname{im} d_{i+1} = \frac{H_i(X^i)}{\operatorname{im}\partial_{i+1}} = H_i(X).$$

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