Homology of some Surfaces

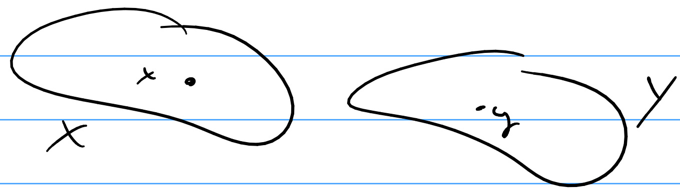
first: know that $H_n(X \sqcup Y) = H_n(X) \oplus H_n(Y)$

not surprisingly: if $A \subset X$ and $B \subset Y$
then $H_n(X \sqcup Y, A \sqcup B) = H_n(X, A) \oplus H_n(Y, B)$

if we have "good" basepoints $x \in X$, $y \in Y$

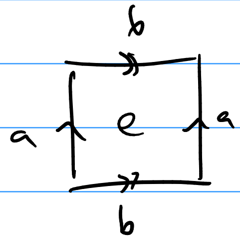
then $\tilde{H}_n(X \vee Y) = \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$

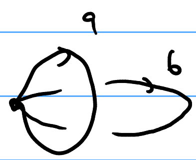
$$\begin{aligned} &\tilde{H}_n(X \vee Y) \\ &\quad \| \\ &H_n(X \vee Y, pt) = H_n(X \sqcup Y, x \sqcup y) = H_n(X, x) \oplus H_n(Y, y) \end{aligned}$$



H_* of torus and Klein bottle w/ coeffs in $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_2$

(know that $\tilde{H}_*(S^n) = \mathbb{Z}$ in deg n, or $\mathbb{Q}$, or $\mathbb{Z}_2$)

$T =$ 

get $S^1 \vee S^1$ 

$\subset T$

and $T / S^1 \vee S^1 = \text{square}/\text{bdry} = S^2$

think of $T = D^2$ glued to $S^1 \vee S^1$
using the map $\partial D^2 = S^1 \to S^1 \vee S^1$
indicated by the picture.

l.e.s of the pair

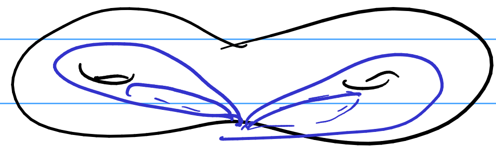
$$\tilde{H}_*(S^1 \vee S^1) \to \tilde{H}_*(T) \to H_*(T, S^1 \vee S^1) = \tilde{H}_*(S^2)$$

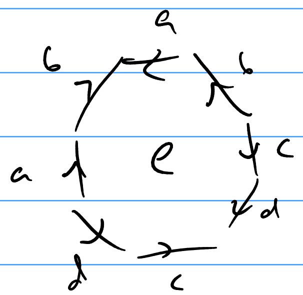
$$\begin{array}{llllll}
\tilde{H}_2 & 0 & \to & \tilde{H}_2(T) & \xrightarrow{\cong} & \mathbb{Z}e \quad \partial = 0 \qquad e \mapsto a+b-a-b = 0 \\
\tilde{H}_1 & \mathbb{Z}a \oplus \mathbb{Z}b & \xrightarrow{\cong} & \tilde{H}_1(T) & \to & 0 \\
\tilde{H}_0 & 0 & \to & \tilde{H}_0(T) & \to & 0
\end{array}$$

$$H_i(T) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 1 \\ \mathbb{Z} & i = 2 \end{cases}$$

or $\mathbb{Q}, \mathbb{Q}^2, \mathbb{Q}$ or $\mathbb{Z}_2, \mathbb{Z}_2^2, \mathbb{Z}_2$ ...

similarly, for a 2-holed torus





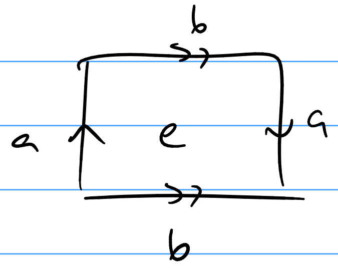
$$S^1 \vee S^1 \vee S^1 \vee S^1 \subset X \qquad X / S^1 \vee S^1 \vee S^1 \vee S^1 \cong S^2$$

$$\begin{array}{lllll}
0 & \to & \tilde{H}_2(X) & \xrightarrow{\cong} & \mathbb{Z} \quad 0 \qquad e \mapsto a+b-a-b+c+d-c-d = 0 \\
\mathbb{Z}^4 & \xrightarrow{\cong} & \hat{H}_1(X) & \to & 0 \\
0 & \to & \hat{H}_0(X) & \to & \mathbb{Z}
\end{array}$$

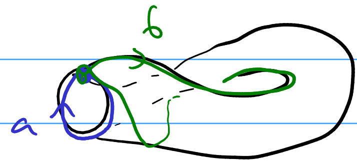
$$H_*(X) = \mathbb{Z}, \mathbb{Z}^4, \mathbb{Z} \quad \text{or} \quad \mathbb{Q}, \mathbb{Q}^4, \mathbb{Q}$$

or $\mathbb{Z}_2, \mathbb{Z}_2^4, \mathbb{Z}_2$

Klein bottle?



$S^1 \vee S^1 \subset K$



$$\tilde H_*(S^1\vee S^1) \to \tilde H_*(K) \to H_*(K, S^1 \vee S^1)$$

$$\begin{array}{llllll}
H_2 & 0 & \longrightarrow & \tilde H_2(K) & \longrightarrow & \mathbb{Z}e \qquad e \mapsto a+b+a-b \\
 & & & & & \qquad\qquad = 2a \\
H_1 & \mathbb{Z}a \oplus \mathbb{Z}b & \longrightarrow & \tilde H_1(K) & \longrightarrow & 0 \\
H_0 & 0 & \longrightarrow & \tilde H_0(K) & \longrightarrow & 0
\end{array}$$

$$\ker\left(\mathbb{Z} \xrightarrow[\binom{2}{0}]{} \mathbb{Z}^2\right) = 0$$

$$\operatorname{coker} = \mathbb{Z}_2 \oplus \mathbb{Z} \qquad \frac{\mathbb{Z}\cdot a \oplus \mathbb{Z}b}{\mathbb{Z}\cdot 2a \oplus 0}$$

$$H_i(K) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_2 \oplus \mathbb{Z} & i=1 \\ 0 & i=2 \end{cases}$$

with $\mathbb{Q}$ coeffs:

$$\begin{array}{lllll}
0 & \to & \tilde H_2 & \to & \mathbb{Q} \quad e \mapsto 2a \ \binom{2}{0} \\
\mathbb{Q}^2 & \to & \tilde H_1 & \to & 0 \\
0 & \to & \tilde H_0 & \to & 0
\end{array}$$

$$\mathbb{Q} \xrightarrow[\binom{2}{0}]{} \mathbb{Q}^2$$

$\ker = 0 \qquad \operatorname{coker} = \mathbb{Q}$

with $\mathbb{Z}_2$ coeffs

$$\begin{array}{lllll}
0 & \to & \tilde H_2 & \to & \mathbb{Z}_2 \quad e \mapsto 2a = 0 \\
\mathbb{Z}_2^2 & & \tilde H_1 & & 0 \\
0 & & \tilde H_0 & & 0
\end{array}$$

$$\mathbb{Z}_2 \xrightarrow[\binom{0}{0}]{} \mathbb{Z}_2^2$$

$\ker = \mathbb{Z}_2 \qquad \operatorname{coker} = \mathbb{Z}_2^2$

$$H_i(k; \mathbb{Q}) = \begin{cases} \mathbb{Q} & i=0 \\ \mathbb{Q} & i=1 \\ 0 & i=0 \end{cases} \qquad H_i(k; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & i=0 \\ \mathbb{Z}_2^2 & i=1 \\ \mathbb{Z}_2 & i=2 \end{cases}$$