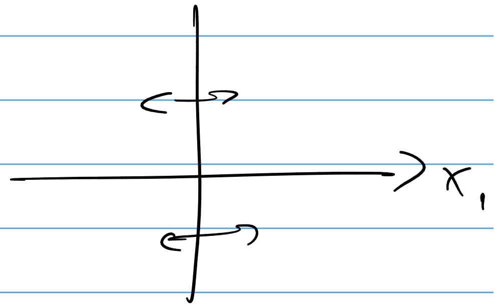
Worksheet last time:

reflection $\quad \rho_n : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$



$$\rho_n : D^n \longrightarrow D^n$$

$$\rho_n : S^{n-1} \longrightarrow S^{n-1}$$

claim: acts by -1 on $H_{n-1}(S^{n-1}) = \mathbb{Z}$

corollary: antipodal map $a : S^n \longrightarrow S^n$

$$v \longmapsto -v$$

induces $(-1)^{n+1}$ on $H_n(S^n) = \mathbb{Z}$

we know that $a \simeq 1$ when n is odd.

$$F : S^{2m-1} \times I \longrightarrow S^{2m-1}$$

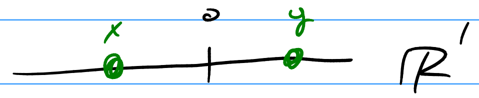
$$F(z_1, \dots, z_m, t) = e^{i\pi t}(z_1, \dots, z_m)$$

$z_i \in \mathbb{C}$

$$F(-, 0) = \mathrm{id} \qquad F(-, 1) = a$$

but this proves $a \not\simeq 1$ when n is even.

Do the claim first for $p_1 : S^0 \longrightarrow S^0$



S^0 = two points

$p_1 \circ x = y \qquad p_1 \circ y = x$

induced map on $H_0 = \mathbb{Z}^2$?

$x : \Delta^0 \longrightarrow S^0$ const map at -1
y : const map at 1

$$\cdots \longrightarrow C_3(S^0) \xrightarrow{\partial} C_2(S^0) \xrightarrow{\partial} C_1(S^0) \xrightarrow{\partial} C_0(S^0) \longrightarrow 0$$

$$\longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & \\ & 1\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}0 & \\ & 0\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}1 & \\ & 1\end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}0 & \\ & 0\end{pmatrix}} \mathbb{Z}^2 = \mathbb{Z}x \oplus \mathbb{Z}y$$

etc. $\quad H_3 = 0 \qquad H_2 = 0 \qquad H_1 = 0 \qquad p_{1\#} \qquad H_0 = \mathbb{Z}^2 \qquad p_{1\#} \qquad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

$C_0(S^0) = \mathbb{Z}^2 = \mathbb{Z}x \oplus \mathbb{Z}y$

l.e.s. of the pair (D^1, S^0)

$$\begin{tikzcd}
H_1(S^0) \arrow[r] & H_1(D^1) \arrow[r] \arrow[d, "p_*"'] & H_1(D^1, S^0) \arrow[r] \arrow[d, "p_*"'] & H_0(S^0) \arrow[r] \arrow[d, "p_*"'] & H_0(D^1) \arrow[r] \arrow[d, "p_*"'] & H_0(D^1, S^0) \arrow[r] & 0 \\
H_1(S^0) \arrow[r] & H_1(D^1) \arrow[r] & H_1(D^1, S^0) \arrow[r] & H_0(S^0) \arrow[r] & H_0(D^1) \arrow[r] & H_0(D^1, S^0) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix}1\\-1\end{pmatrix}"] \arrow[d, "-1"] & \mathbb{Z}^2 \arrow[r, "(1\ 1)"] \arrow[d, "\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}"] & \mathbb{Z} \arrow[d, "1"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & \mathbb{Z}^2 \arrow[r, "(1\ 1)"'] & \mathbb{Z}
\end{tikzcd}$$

$c \longmapsto (c, -c) \qquad (a, b) \longmapsto a + b$

$-c \longmapsto (-c, c) \qquad (a,b) \mapsto (b,a)$

$i : S^0 \hookrightarrow D^1$
$i \circ x$ and $i \circ y$ are different in $C_0(D^1)$ but the same in $H_0(D^1)$

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = (-1)\begin{pmatrix}1\\-1\end{pmatrix}$$

also said that the identification $D^1/S^0 \cong S^1$ works with our reflections.

$$\begin{tikzcd} D^1/S^0 \arrow[r, "\cong"] \arrow[d, "\rho_1"'] & S^1 \arrow[d, "\rho_2"] \\ D^1/S^0 \arrow[r, "\cong"] & S^1 \end{tikzcd}$$

Iou: $H_1(D^1, S^0) \to H_1(D^1/_{S^0}, pt)$ is an iso.

accept this $\leadsto$ $\rho_{2*} : H_1(S^1) \to H_1(S^1)$ is -1.

from S^1 to S^2

$$\begin{tikzcd} H_2(D^2) \arrow[r] \arrow[d, "\rho_{2*}"'] & H_2(D^2, S^1) \arrow[r] \arrow[d, "\rho_{2*}"'] & H_1(S^1) \arrow[r] \arrow[d, "\rho_{2*}"'] & H_1(D^2) \arrow[d, "\rho_{2*}"'] \\ H_2(D^2) \arrow[r] & H_2(D^2, S^1) \arrow[r] & H_1(S^1) \arrow[r] & H_1(D^2) \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, hook, two heads, "\cong"] \arrow[d, "-1"'] & \mathbb{Z} \arrow[r] \arrow[d, "-1"] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, hook, two heads, "\cong"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

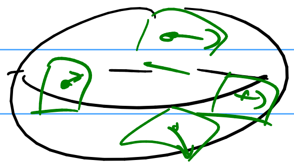
$\rho_{3*} : H_2(S^2, pt) \to H_2(S^2, pt)$ is -1

Hairy ball thm: can't comb the hair on S^2 without a cowlick.

$\nexists$ non-vanishing vector field on S^2

a vector field on S^2 is a map

$v: S^2 \to \mathbb{R}^3$ with $v(x) \perp x$ $\forall x \in S^2$



Proof: suppose that $v: S^2 \to \mathbb{R}^3$ were a non-vanishing v.f.

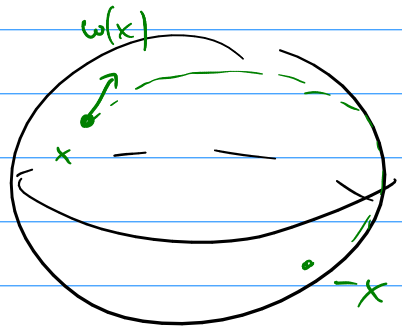
let $w(x) = \dfrac{v(x)}{|v(x)|}$ ← used non-vanishing here. So $w: S^2 \to S^2$ and $w(x) \perp x$ $\forall x \in S^2$

Define $F: S^2 \times I \to S^2$

$(x, t) \longmapsto \cos \pi t \cdot x + \sin \pi t \cdot w(x)$

then $F(x, 0) = x$ identity
$F(x, 1) = -x$ antipodal map.

$\rightarrow \leftarrow$ $\boxtimes$



Other applications of $H_*(S^n)$:

<u>Cor.</u> S^n is not contractible.

because $H_n(S^n) = \mathbb{Z}$ $\quad (n \geq 1)$

<u>Cor.</u> $\mathbb{R}^n \not\cong \mathbb{R}^m$ if $n = m$

Pf: a homeomorphism $f: \mathbb{R}^n \to \mathbb{R}^m$ would induce a homeo

$$f: \mathbb{R}^n \setminus 0 \longrightarrow \mathbb{R}^m \setminus f(0)$$

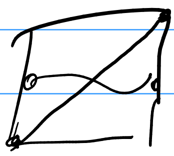
hence a homotopy equiv

$$S^{n-1} \longrightarrow S^{m-1}$$

but they have different homology groups. $\square$

<u>Cor.</u> Brouwer Fixed-Point Theorem:

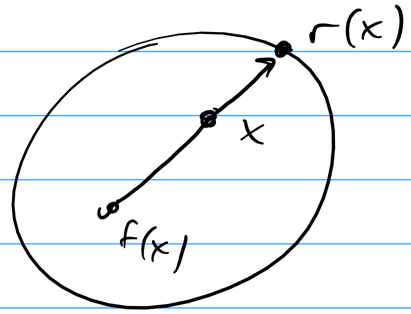
every cont. $f: D^n \to D^n$ has a fixed point

for D^2, we used π_1,
for D^1, just use intermediate value thm: 

as before, if f had no fixed point,

construct $r: D^n \to S^{n-1}$ with

$$r|_{S^{n-1}} = id.$$

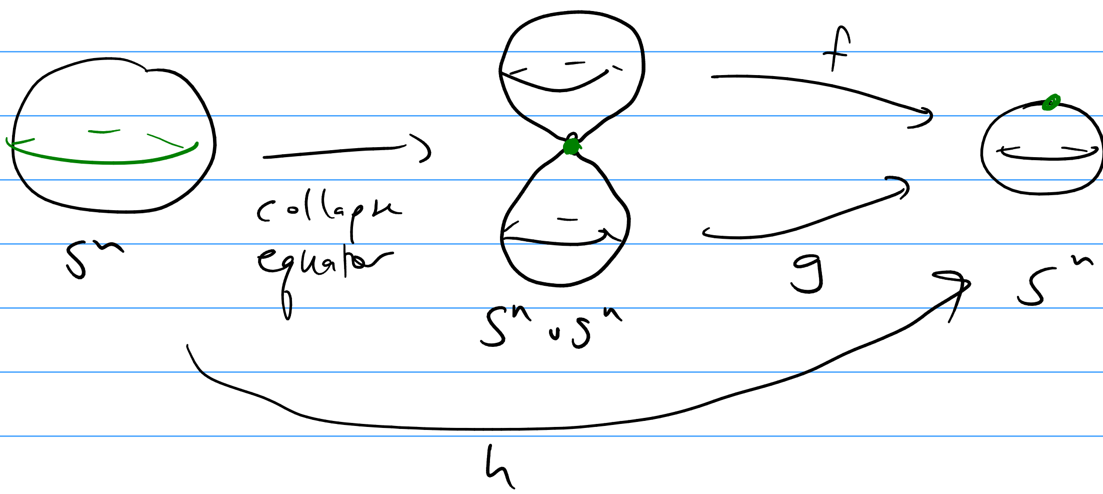


$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "i"] \arrow[rr, bend right, "1"'] & D^n \arrow[r, "r"] & S^{n-1} \end{tikzcd}$$

take H_{n-1}:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[rr, bend right, "1"'] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$ no good. $\boxed{\lightning}$

we'll need soon:



Want to know: $h_*: H_n(S^n) \longrightarrow H_n(S^n)$

$h_* = f_* + g_*$ think about it...