A chain complex:

$$A^{\cdot} = \quad \cdots \xrightarrow{\partial} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots$$

$$\partial^2 = 0$$

$$H_n(A) = \ker \partial_n / \text{image } \partial_{n+1}$$

A map of these:

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"] \arrow[dl, "P_n"'] & A_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"] \arrow[dl, "P_{n-1}"'] & \cdots \arrow[dl, "P_{n-2}"'] \\
\cdots \arrow[r] & B_n \arrow[r, "\partial_n"] & A_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

φ commutes with ∂. φ induces a map $\varphi_* \, H_n(A) \to H_n(B)$

A chain homotopy:
if $\varphi = P\partial + \partial P$ then $\varphi_* = 0$.

A short exact seq. of chain complexes

$$0 \to A_\cdot \xrightarrow{\varphi} B_\cdot \xrightarrow{\psi} C_\cdot \to 0$$

meaning

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r, "\varphi_n"] \arrow[d, "\partial"] & B_n \arrow[r, "\psi_n"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\varphi_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

if $[z] \in H_n(C)$

$y \mapsto z$

$x \mapsto \partial y \mapsto 0$

get a long exact sequence

$$\cdots \to H_n(A_\cdot) \xrightarrow{\varphi_*} H_n(B_\cdot) \xrightarrow{\psi_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{\varphi_*} H_{n-1}(B) \to \cdots$$

$$[z] \longmapsto [x]$$

Naturality: if

$$\begin{tikzcd} 0 \arrow[r] & A_\cdot \arrow[r, "\varphi"] \arrow[d, "f"] & B_\cdot \arrow[r, "\psi"] \arrow[d, "g"] & C_\cdot \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A_\cdot' \arrow[r, "\varphi'"] & B_\cdot' \arrow[r, "\psi'"] & C_\cdot' \arrow[r] & 0 \end{tikzcd}$$

commutes

then this commutes

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A_\cdot) \arrow[r, "\varphi_*"] \arrow[d, "f_*"] & H_n(B_\cdot) \arrow[r, "\psi_*"] \arrow[d, "g_*"] & H_n(C_\cdot) \arrow[r, "\partial"] \arrow[d, "h_*"] & H_{n-1}(A_\cdot) \arrow[r, "\varphi_*"] \arrow[d, "f_*"] & H_{n-1}(B_\cdot) \cdots \arrow[d, "g_*"] \\ \cdots \arrow[r] & H_n(A_\cdot') \arrow[r, "\varphi_*'"] & H_n(B_\cdot') \arrow[r, "\psi_*'"] & H_n(C_\cdot') \arrow[r, "\partial"] & H_{n-1}(A_\cdot') \arrow[r, "\varphi_*'"] & H_{n-1}(B_\cdot') \cdots \end{tikzcd}$$

in our context, if $A \overset{i}{\subset} X$ and $B \overset{j}{\subset} Y$ and $f: X \to Y$ with $f(A) \subset B$

then I get

$$\begin{tikzcd} H_n(A) \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(X) \arrow[r, "q_*"] \arrow[d, "f_*"] & H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"] & H_{n-1}(A) \arrow[r] \arrow[d, "f_*"] & \cdots \\ H_n(B) \arrow[r, "j_*"] & H_n(Y) \arrow[r, "q_*'"] & H_n(Y,B) \arrow[r, "\partial"] & H_{n-1}(B) & \end{tikzcd}$$

Thinking about l.e.s. of pair geom. $A \subset X$

$$H_n(A) = \text{n-chains in } A \text{ with } \partial = 0 \;/\; \text{boundaries of } n+1 \text{ chains in } A$$

$$\downarrow i_* \qquad\qquad \downarrow i_\# \qquad\qquad \downarrow i_\#$$

$$H_n(X) \quad \text{n-chains in } X \text{ with } \partial = 0 \;/\; \text{boundaries of } n+1 \text{ chains in } X$$

$$\downarrow j_* \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$H_n(X,A) \quad \text{n-chains in } X \text{ with } \partial \subset A \;/\; \text{boundaries of } n+1 \text{ chains in } X + \text{n-chains in } A \xrightarrow{\partial} 0$$

$$\downarrow \partial \qquad\qquad \downarrow \partial \qquad\qquad \downarrow$$

$$H_{n-1}(A) = \text{(n-1)-chains in } A \text{ with } \partial = 0 \;/\; \text{boundaries of } n \text{ chains in } A$$

$$\downarrow i_* \qquad\qquad \downarrow i_\# \qquad\qquad \downarrow i_\#$$

$$H_{n-1}(X) \quad \text{(n-1)-chains in } X \text{ with } \partial = 0 \;/\; \text{boundaries of } n \text{ chains in } X$$

im j_* ⊂ ker i_* ker i_* ⊂ im j_*

im j_* ⊂ ker ∂ because $\partial^2 = 0$ ker ∂ ⊂ im j_* (hardest)

im ∂ ⊂ ker i_* ker i_* ⊂ im ∂