Reading: Hatcher has a comment abt. l.e.s. of pair in <u>reduced</u> homology

$$\tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_{n-1}(A) \to \cdots$$

$\tilde{H}_n(X,A) := H_n(X,A)$
but don't...

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & & \text{deg } 1 \\
0 \arrow[r] & C_1(A) \arrow[r] \arrow[d] & C_1(X) \arrow[r] \arrow[d] & C_1(X,A) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & C_0(A) \arrow[r] \arrow[d, "\text{aug}"] & C_0(X) \arrow[r] \arrow[d, "\text{aug}"] & C_0(X,A) \arrow[r] \arrow[d] & 0 & \text{deg } 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "1"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 & \text{deg } -1 \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

triple $A \subset B \subset X$

$$C_n(X,B) = \frac{C_n(X)}{C_n(B)} \overset{3^{rd}}{=} \frac{C_n(X)/C_n(A)}{C_n(B)/C_n(A)} = \frac{C_n(X,A)}{C_n(B,A)}$$

$$\begin{tikzcd}
 & \arrow[d, "\partial"] & \arrow[d, "\partial"] & \arrow[d, "\partial"] & \\
0 \arrow[r] & C_n(B,A) \arrow[r] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & C_n(X,B) \arrow[r] \arrow[d, "\partial"] & 0 \\
 & \, & \, & \, &
\end{tikzcd}$$

s.e.s of chain complexes !!
$\rightsquigarrow$ l.e.s. in homology

$$\to H_n(B,A) \to H_n(X,A) \to H_n(X,B) \to$$

LES of a Pair: Proof

$A \subset X$

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i_\#"] \arrow[d, "\partial"'] & C_n(X) \arrow[r, "q"] \arrow[d, "\partial"'] & C_n(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_\#"] & C_{n-1}(X) \arrow[r, "q"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$\alpha \mapsto q\alpha$

$\partial\alpha \mapsto \partial\alpha \mapsto \partial q\alpha = q\,\partial\alpha = 0$

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \to \cdots$$

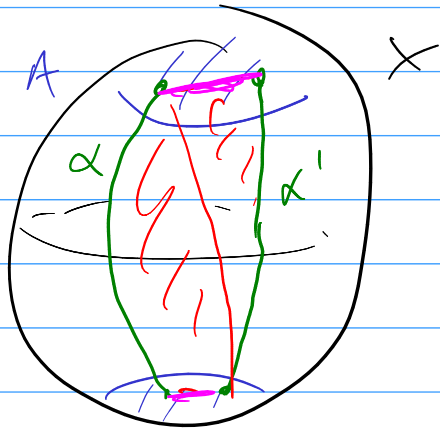
$H_n(A) = \dfrac{\text{cycles in } A}{\text{boundaries}}$, $H_n(X) = \dfrac{\text{cycles in } X}{\text{boundaries}}$, $\boxed{H_n(X,A) = \dfrac{\text{chains in } X \text{ with bdry in } A}{\text{bdries in } X + \text{anything in } A}}$

∂ takes the boundary:

in the example,
$[q\alpha] \in H_1(X,A)$

$\partial[q\alpha] = [\partial\alpha] \in H_0(A)$

$=$ 



$\alpha \in Z_1(X,A)$
$\subset C_1(X,A)$

is ∂ well-defined?

given $c, c' \in C_n(X)$
with $\partial c, \partial c' \in C_{n-1}(A) \subset C_{n-1}(X)$
(so $qc, qc' \in Z_n(X,A)$
so they rep. classes in $H_n(X,A)$)

if they rep. the same class in
$H_n(X,A)$ then

$c - c' = \partial b + a$ where $b \in C_{n+1}(X)$
and $a \in C_n(A)$

$\alpha - \alpha' \in B_1(X,A)$

$\beta \in C_2(X)$

$\partial\beta = \alpha - \alpha' +$ 2 thing in A

so $\partial(q\beta) = q(\partial\beta) = q\alpha - q\alpha' + 0$

so $[q\alpha] = [q\alpha'] \in H_1(X,A)$

So $\partial c - \partial c' = \cancel{\partial\partial b}^{\,0} + \partial a$

So $[\partial c] = [\partial c']$ in $H_{n-1}(A)$.

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "i_\#"] \arrow[d, "\partial"] & C_n(X) \arrow[r, "j"] \arrow[d, "\partial"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "i_\#"] & C_{n-1}(X) \arrow[r, "j"] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$a \longmapsto i_\# a$

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \cdots$$

want: $\operatorname{im} i_* = \ker j_*$, $\operatorname{im} j_* = \ker \partial$, $\operatorname{im} \partial = \ker i_*$

$\frac{3}{2}$ are easy: $\operatorname{im} i_* \subset \ker j_*$

$\operatorname{im} j_* \subset \ker \partial$

$\operatorname{im} \partial \subset \ker i_*$

$\ker(i_*) \subset \operatorname{im} \partial$?

suppose $[a] \in H_{n-1}(A)$

with $i_*[a] = 0$

So $a \in C_{n-1}(A)$ and $i_\# a = \partial c$ for $c \in$.

[my pen is freaking out, let's go to breakout now.]