A seq. of maps of ab. groups or R-modules

$$\cdots \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

is <u>exact</u> at B if $\ker g = \operatorname{im} f$.

notice ① implies $g \circ f = 0$

② if we're exact at B
then $f = 0$ iff g is injective

③ similarly, $g = 0$ iff f is surjective

saying $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact
(everywhere)

means $C \cong B/A$

1st iso thm: $\operatorname{im} g \cong B/\ker g$

Last time: For subspace $A \subset X$,

defined $C_n(X,A) = C_n(X)/C_n(A)$

∂ descends ⇝ get $H_n(X,A) = \ker \partial / \operatorname{im} \partial$

Theorem 1: quotient map $X \to X/A$ induces an iso

$$H_n(X,A) \longrightarrow H_n(X/A, \underbrace{A/A}_{=\text{point}}) \;\; = \tilde{H}_n(X/A) \quad \text{if } A \text{ is "good"}$$

maybe Robert will prove this next term.

Theorem 2: $\exists$ a "connecting homomorphism"

$$\partial : H_n(X,A) \longrightarrow H_{n-1}(A)$$

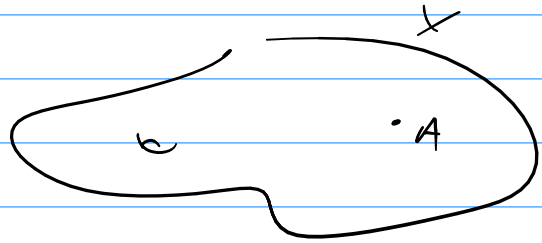
such that the seq.

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to \cdots$$

is exact.

Proof: next week

What can we do with it?



① A = point

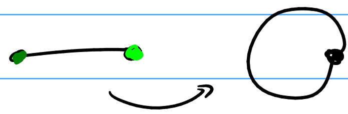
$$0 \to \cancel{H_2(pt)} \to H_2(X) \xrightarrow{\cong} H_2(X,pt) \to$$
$$0 \to \cancel{H_1(pt)} \to H_1(X) \xrightarrow{\cong} H_1(X,pt) \to 0$$
$$\to H_0(pt) \to H_0(X) \to \underline{H_0(X,pt)} \to 0$$

$\mathbb{Z} \longleftrightarrow \mathbb{Z}^{\#\text{ path components}}$ $\qquad$ $\mathbb{Z}^{\#\text{ components}-1}$

Conclude: $H_n(X, \text{point}) = \begin{cases} H_n(X) & \text{if } n \geq 1 \\ \mathbb{Z}^{\#\text{ components}-1} & \text{if } n = 0 \end{cases}$

"reduced homology" $\quad \tilde{H}_n(X)$

② $X = D^1 \qquad A = S^0 \qquad X/A = S^1$



$H_2 \qquad 0 \to 0 \to \cancel{H_2(S^1,pt)} \to$

$H_1 \qquad 0 \to 0 \to \cancel{H_1(S^1,pt)} \to$ $\mathbb{Z}$ $\binom{1}{-1}$

$H_0 \qquad \to \mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z} \to \cancel{H_0(S^1,pt)} \to 0$

0 bec. green map is surjective

$\binom{1}{0} \longmapsto 1$

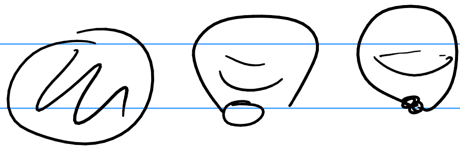
$\binom{0}{1} \longmapsto 1$

$\binom{a}{b} \longmapsto a+b$

$\ker = \left\{ \binom{a}{-a} \;\middle|\; a \in \mathbb{Z} \right\} \cong \mathbb{Z}$

thus $H_n(S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} & n=1 \\ 0 & \text{otherwise} \end{cases}$

③ $A = S^1 \quad X = D^2 \quad X/A = S^2$



$$\begin{tikzcd}
H_3 & 0 \arrow[r] & 0 \arrow[r] & \cancel{H_3(S^2, pt)} \, 0 \arrow[dll, bend left] \\
H_2 & 0 \arrow[r] & 0 \arrow[r] & H_2(S^2, pt) \arrow[dll, "\cong"] \\
H_1 & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \cancel{H_1(S^2, pt)} \, 0 \arrow[dll, "0"] \\
H_0 & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r, "0"] & H_0(S^2, pt) \arrow[r] & 0 \\
 & A & X & (X, A)
\end{tikzcd}$$

So $H_n(S^2) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & n > 2 \end{cases}$