Last time: given $f \simeq g : X \to Y$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[r] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] \arrow[dl, "P"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"'] & C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$$g_\# - f_\# = P\partial + \partial P$$

worksheet: conclude that $f_* = g_* : H_n(X) \to H_n(Y)$

given $[z] \in H_n(X)$ represented by $z \in C_n(X)$ with $\partial z = 0$

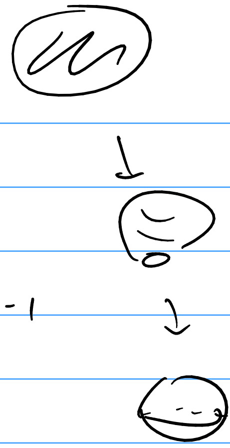
we have $g_*([z]) := [g_\#(z)]$

$g_\# z = f_\# z + \cancel{P\partial z} + \partial P z$ — this is a boundary!

so $[g_\# z] = [f_\# z]$

We'd like to compute $H_*(S^n)$.

Do it inductively, by viewing $S^n = D^n / S^{n-1}$



Start with $S^0 = \cdot \quad \cdot$

<u>Prop</u>: $H_n(X \sqcup Y) = H_n(X) \oplus H_n(Y)$

<u>Pf</u>. Bec. Δ^n is connected,

a map $\sigma : \Delta^n \longrightarrow X \sqcup Y$
lands in either X or Y.

so $C_n(X \sqcup Y) = C_n(X) \oplus C_n(Y)$
∂ respects this decomp.
so $Z_n(X \sqcup Y) = Z_n(X) \oplus Z_n(Y)$
$B_n(X \sqcup Y) = B_n(X) \oplus B_n(Y)$

conclusion follows $\square$

$$H_n(S^0) = H_n(2 \text{ points}) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

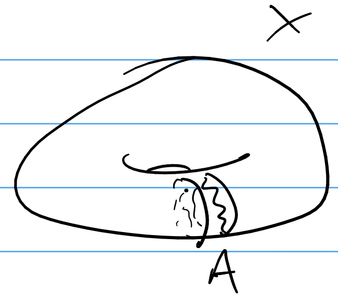
<u>also</u> if X has k path components

then $H_0(X) = \mathbb{Z}^k$

To talk about $H_*(X/A)$, need

Relative Homology

If $A \subset X$, then $C_n(A) \subset C_n(X)$



Define $C_n(X,A) = C_n(X)/C_n(A)$

So a rel. chain is represented by
a chain in X,
and two chains in X rep. the same rel. chain
if their diff. is a chain in A.

The map $\partial: C_n(X) \to C_{n-1}(X)$

$i: A \hookrightarrow X$

takes $C_n(A)$ into $C_{n-1}(A)$

so induces $C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A)$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & C_n(A) \arrow[r, "\partial"] \arrow[d, "i_*"] & C_{n-1}(A) \arrow[r] \arrow[d, "i_*"] & \cdots \\
\cdots \arrow[r] & C_n(X) \arrow[r, "\partial"] \arrow[d, "q"] & C_{n-1}(X) \arrow[r] \arrow[d, "q"] & \cdots \\
\cdots \arrow[r] & C_n(X,A) \arrow[r, "\partial"] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

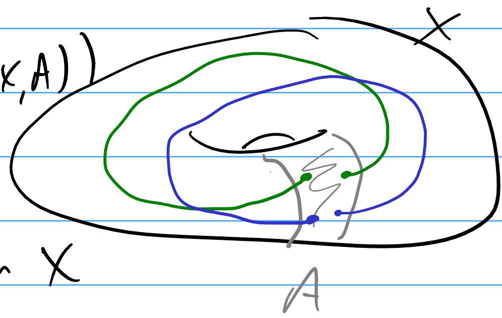
So relative cycles $:= \ker\left(\partial : C_n(X,A) \to C_{n-1}(X,A)\right)$

are represented by chains in X with boundary in A.

$c \in C_n(X)$, $\quad q(c) \in C_n(X,A)$

if $\partial q(c) = 0$ then $q(\partial c) = 0$ so $\partial c \in C_n(A)$

Relative boundaries $:= \operatorname{im}\left(\partial : C_{n+1}(X,A) \to C_n(X,A)\right)$



are represented by any chain in A + bdry of a chain in X

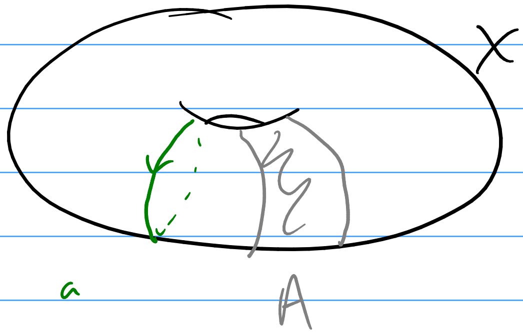
given $c \in C_{n-1}(X)$, if qc is a boundary

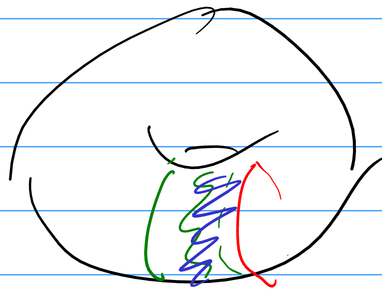
then $qc = \partial q c' = q \partial c'$ for some $c' \in C_n(X)$

then $q(c - \partial c') = 0$ so $c - \partial c' \in C_n(A)$

define $H_n(X,A) = $ rel. cycles / rel. boundaries.

one more:





in X, a is a cycle but not a boundary.

it is a relative boundary though.

Two theorems will let us compute:

1. let $f: X \to X/A$ be the quot map.

   if the subspace $A \subset X$ is "good"

   then $f_*: H_n(X,A) \to H_n(X/A, pt)$

   is an iso.

2. We'll define a map $\partial: H_n(X,A) \to H_{n-1}(A)$

   chain in X w/ bdry in A $\mapsto$ its bdry

   This sequence is exact:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "i_*"] & H_n(X) \arrow[r, "q_*"] & H_n(X,A) \arrow[dll, "\partial"'] & \\
& H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(X) \arrow[r] & \cdots
\end{tikzcd}$$