Induced Maps + Homotopy Invariance

Last time:

chains $C_n(X)$ = formal linear combos of maps $\sigma : \Delta^n \to X$

differentials $\cdots \to C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$

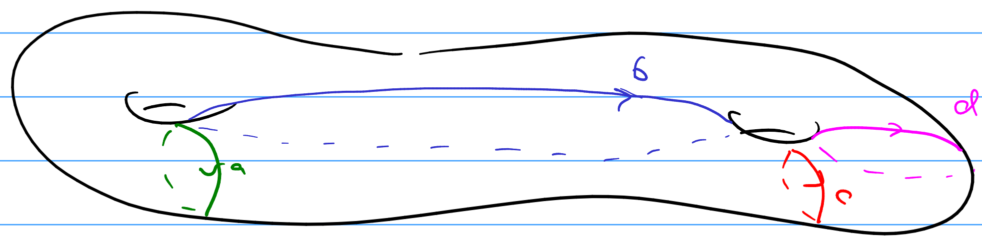
with $\partial_n \circ \partial_{n+1} = 0$

thus $\operatorname{im} \partial_{n+1} \subset \ker \partial_n$

boundaries $\subset$ cycles

$H_n(X) = \ker \partial_n / \operatorname{im} \partial_{n+1}$ tends to be fin. gen.

Elements of H_n are represented by n-cycles



two cycles represent the same class in H_n if they differ by a boundary. ("homologous")

$[c] = [d]$ $\quad$ $[a] = [b] + [c]$

First calculations:

Prop. If X is path-conn. then $H_0(X) = \mathbb{Z}$

Pf View a point $x_0 \in X$ as a 0-chain

Then $\partial x_0 = 0$ so $[x_0] \in H_0(X)$

if $x_1 \in X$ is another point,
choose a path γ from x_0 to x_1
View γ as a 1-chain
Then $\partial \gamma = x_1 - x_0$

so $[x_1] = [x_0]$ in H_0.

a gen. elt. of H_0 is $a_0[x_0] + a_1[x_1] + \dots + a_k[x_k]$
$a_i \in \mathbb{Z}$

But this $= (\Sigma a_i)[x_0]$

so $H_0(X) = \mathbb{Z}$. $\boxed{\checkmark}$

Prop $H_n(\text{point}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$

Pf $\exists$ only one map $\sigma_n : \Delta^n \to \text{point}$

$$\partial \sigma_n = \sum_{i=0}^{n} (-1)^i \sigma_n \circ \varphi_i = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} \sigma_{n-1} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

So $\cdots \to C_3 \to C_2 \to C_1 \to C_0 \to 0$ is

$$\cdots \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

$H_0(pt) = \mathbb{Z}/0 = \mathbb{Z}$

$H_1 = \mathbb{Z}/\mathbb{Z} = 0$

$H_2 = 0/0 = 0$

$H_3 = \mathbb{Z}/\mathbb{Z} = 0$

etc. $\boxed{\mathbb{Z}}$

Suppose $f : X \to Y$.

Define $f_\# : C_n(X) \to C_n(Y)$ and extend linearly.
$\sigma \longmapsto f \circ \sigma$

Lemma $\partial f_\# \sigma = f_\# \partial \sigma$ for all $\sigma : \Delta^n \to X$ hence for all chains.

Pf $\partial f_\# \sigma = \partial(f \circ \sigma) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ \varphi_i$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$

$f_\# \partial \sigma = f_\# \left(\sum_{i=0}^{n} (-1)^i \sigma \circ \varphi_i \right) = \sum_{i=0}^{n} (-1)^i f_\# (\sigma \circ \varphi_i)$ $\square$

As a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Thus $f_\#(Z_n(X)) \subset Z_n(Y)$

if $\partial c = 0$ then $\partial(f_\# c) = f_\#(\partial c) = 0$

and $f_\#(B_n(X)) \subset B_n(Y)$

so we get a well-defined map

$$f_*: H_n(X) \longrightarrow H_n(Y).$$

Easy to check: $(g \circ f)_* = g_* \circ f_*$

$$1_* = 1$$

Thus homeomorphic spaces have isomorphic H_ns.

Want: homotopy invariance.

Prop if $f \simeq g : X \to Y$

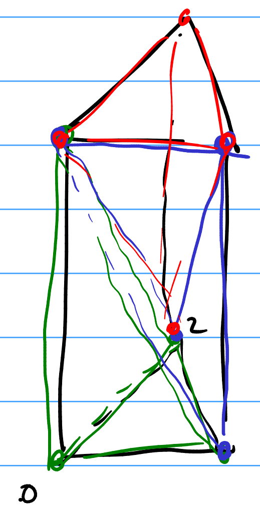
then $f_* = g_* : H_n(X) \to H_n(Y)$

Pf. let $F : X \times I \to Y$ be a homotopy from f to g

Define $P : C_n(X) \to C_{n+1}(Y)$ as follows "prism"

Given $\sigma : \Delta^n \to X$, consider

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times I \xrightarrow{\sigma \times 1} X \times I \xrightarrow{F} Y$$

$\Delta^3 \longrightarrow$  $\Delta^2 \times I$

p_0

p_1

p_2

Define $P(\sigma) = \sum (-1)^i F \circ (\sigma \times 1) \circ p_i$ and extend linearly.

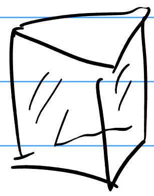
I claim that $\partial P + P\partial = g_\# - f_\#$:

for $\sigma : \Delta^n \to X$,

$\partial P\sigma$ is

sides, top, bottom

$P\partial\sigma$ is

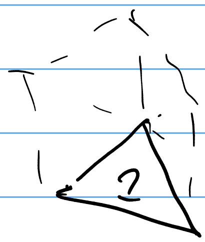


just the sides.

$g_\# \sigma$ is

just the top

$f_\# \sigma$ is



bottom, opp. orientation.

worksheet: conclude that $f_* = g_*$.