Homology — Motivation + Defs.

Understand π_1: big thing / big equiv. rel

$\rightsquigarrow$ something small
+ computable

But π_1 doesn't see a lot of higher-dim'l info.

e.g. can't tell that S^2 is not contractible.

One generalization:
if $\pi_1 = \{\text{pointed maps } S^1 \to X\} / \text{htpy rel. basepoint}$

then $\pi_n = \{\text{pointed maps } S^n \to X\} / \text{htpy rel. basepoint}$

this is an Abelian group (not obvious)

hard to compute — no Van Kampen thm

flavor: seen that $\pi_n(S^1) = \begin{cases} 0 & n=0 \text{ (path-conn)} \\ \mathbb{Z} & n=1 \\ 0 & n\geq 2 \end{cases}$

but

n	0	1	2	3	4	5	6	7	8	9
$\pi_n(S^2)$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$

10	11	12	13	14	15	
$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{84}\times\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	....

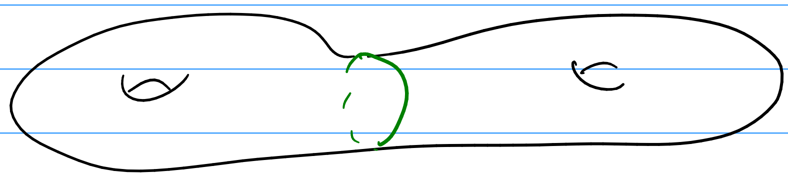
Could also say

$$\pi_1(X) = \{\text{loops in } X\} / \{\text{loops that bound a disc}\}$$

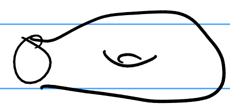
could consider more generally

$$\{\text{loops in } X\} / \{\text{loops that bound a surface}\}$$

e.g. this loop



doesn't bound a disc
but does bound



one interpretation of this idea
$\rightsquigarrow$ cobordism

another interpretation $\rightsquigarrow$ homology

Def the standard n-simplex

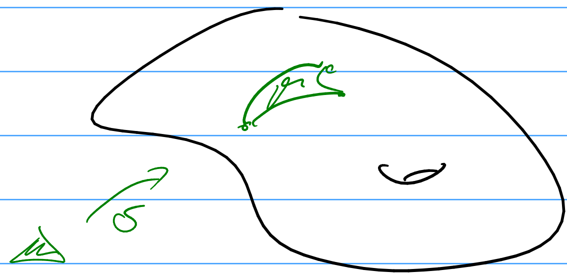
$$\Delta^n = \{ (x_0, \dots x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \ \sum x_i = 1 \}$$

Δ_3 is a tetrahedron...

An <u>n-simplex</u> in a space X is a map

$$\sigma : \Delta^n \longrightarrow X$$

maybe injective, maybe not.



An <u>n-chain</u> is a formal $\mathbb{Z}$-linear combination of n-simplices

e.g. $\sigma_1 + 2\sigma_2 - 5\sigma_3$

they form an ab. group $C_n(X)$

could take coeffs in an ab. group A or a ring R

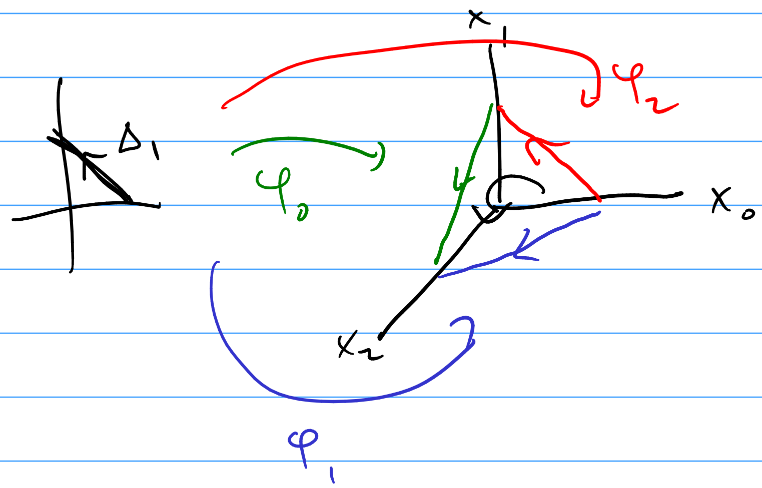
e.g. $C_n(X, \mathbb{Q})$ = $\mathbb{Q}$-linear combos of simplices

Face <u>maps</u> $\varphi_i : \Delta^{n-1} \longrightarrow \Delta^n \qquad i = 0, 1, \ldots, n$

$$(x_0, \ldots, x_{n-1}) \longmapsto (x_0, \ldots, x_{i-1}, \underset{\uparrow}{0}, x_i, \ldots x_{n-1})$$

i^{th} place

e.g. $\varphi_1, \varphi_2, \varphi_3 : \Delta^1 \longrightarrow \Delta^2$



If $\sigma : \Delta^n \to X$ is an n-simplex,

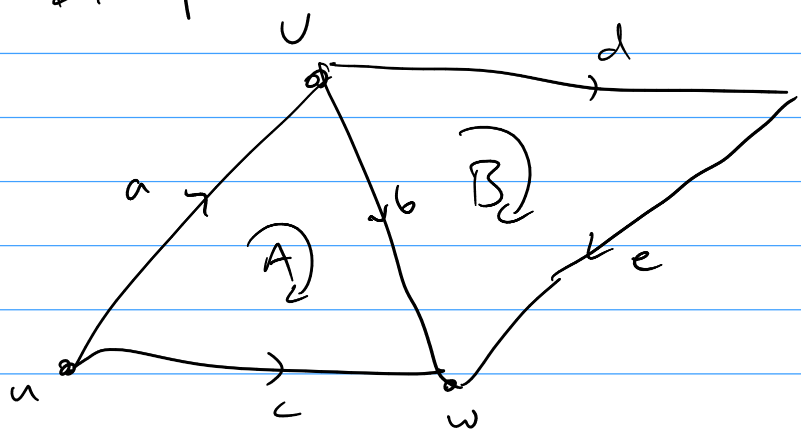
define the <u>boundary</u> $\partial\sigma = \sum_{i=0}^{n} (-1)^i \underline{\sigma \circ \varphi_i} \in C_{n-1}(X)$

Extend linearly to get a map

$$C_n(X) \xrightarrow{\partial} C_{n-1}(X)$$

i.e. $\partial(\sigma_1 + 2\sigma_2 - 5\sigma_3) = \partial\sigma_1 + 2\partial\sigma_2 - 5\partial\sigma_3$

Example:



$\partial A = a + b - c$
$\partial B = d + e - b$

$\partial(A+B) = a + d + e - c$

$\partial a = v - u$
$\partial b = w - v$
$\partial c = w - u$

$\partial(\partial A) = (v-u) + (w-v) - (w-u) = 0$

Lemma: for any n-simplex $\sigma : \Delta^n \to X$, $\partial \partial \sigma = 0$ (worksheet)

thus

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[rr, bend right, "\partial^2 = 0"'] & C_{n-1}(X) \arrow[r, "\partial"] & C_{n-2}(X) \end{tikzcd}$$

More defs:

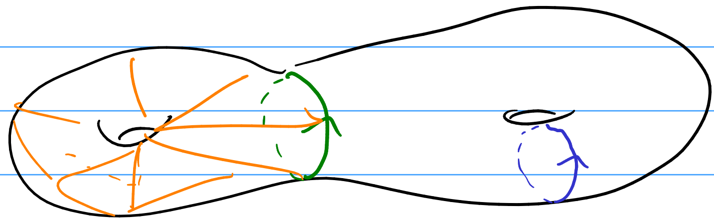
an n-cycle is an n-chain $c \in C_n(X)$ with $\partial c = 0$

$$Z_n := \ker(\partial : C_n \to C_{n-1})$$

an n-boundary is an n-chain of the form ∂c for some $c \in C_{n+1}$

$$B_n := \operatorname{im}(\partial : C_{n+1} \to C_n)$$

example:



both are cycles
only green one is a boundary.

Because $\partial^2 = 0$, we have $B_n \subset Z_n$

Define $H_n(X) = Z_n(X) / B_n(X)$.