Classification of Covers & Universal Cover

Last time, a cover $p: \tilde{X} \to X$ is <u>normal</u>

if $p_* \pi_1(\tilde{X}) \trianglelefteq \pi_1(X)$.

Then intermediate covers $\longleftrightarrow$ intermediate subgroups

Suppose we had a cover $p: \tilde{X} \to X$ with $\pi_1(\tilde{X}) = 1$

Then for every other cover $p': X' \to X$,

$$\begin{tikzcd} & X' \arrow[d, "p'"] \\ \tilde{X} \arrow[r, "p"'] \arrow[ur, dashed, "\exists \tilde{p}"] & X \end{tikzcd}$$

then $\mathrm{im}(p_*) \subset \mathrm{im}(p'_*)$

so we get a lift

can show that $\tilde{p}$ is a covering map.

If we choose basepoints, $\exists! \, \tilde{p}$

so $\tilde{X}$ is called the <u>universal</u> cover of X.

<u>Thm.</u> (classification of covers):

Suppose X has a simply conn. cover

Choose a basepoint $x_0 \in X$.

Then covers $p': X' \to X$ with chosen basepoint $x_0' \in p'^{-1}(x_0)$ are in bijection with subgroups of $\pi_1(X, x_0)$

and covers w/o basepoints are in bijection with conj. classes of subgroups.

Pf: Immediate from Galois corresp.
Observe that $1 \trianglelefteq \pi_1(X,x_0)$ and $\mathrm{Aut}(\tilde{X}/X) = \dfrac{\pi_1(X,x_0)}{1}$ ☐

Examples of universal covers:

(1) $\mathbb{R} \to S^1 \qquad x \mapsto e^{2\pi i x}$

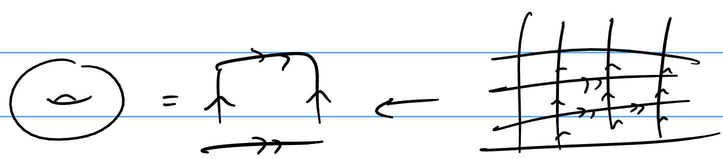
(2) $S^n \xrightarrow{2:1} \mathbb{RP}^n$

(3) surfaces?

S^2 is simply conn. so it's its own univ. cover.

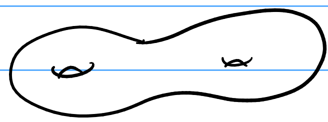
$\mathbb{RP}^2 \leftarrow S^2$

torus $= S^1 \times S^1 \leftarrow \mathbb{R}^2$



Klein bottle $\xleftarrow{2:1}$ torus $\leftarrow \mathbb{R}^2$

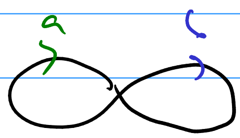
higher genus?

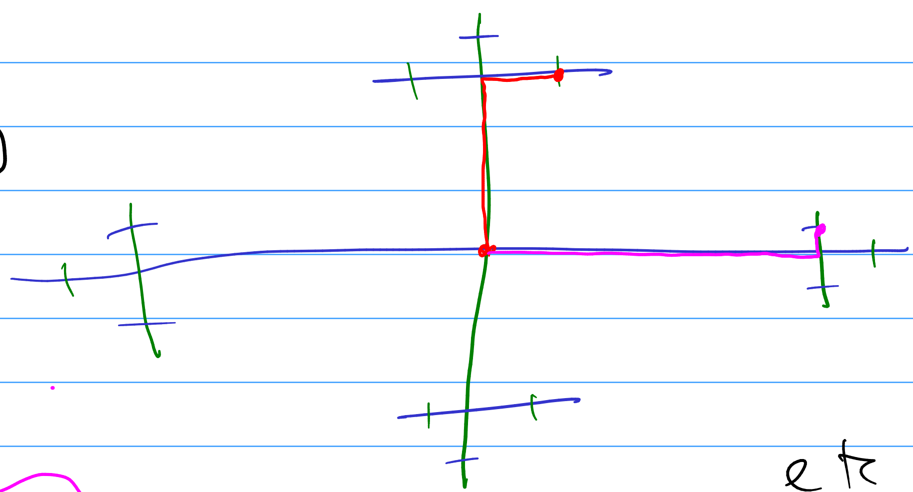


again $\mathbb{R}^2$ but not obvious.

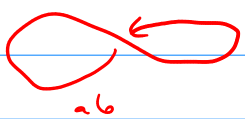
worksheet ..

(4) $S^1 \vee S^1$

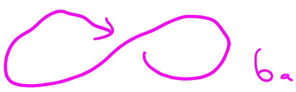




$ab \neq ba$



see it via lifts:



etc.

Thm: Suppose X is connected, loc. path-conn, and loc. simply-connected. Then X has a simply-connected cover.

(Semi-loc. simply connected is enough.

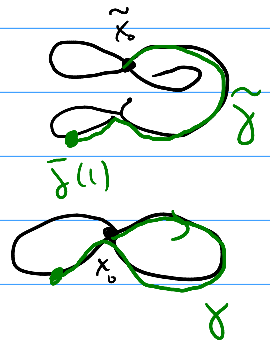
Equivalently, $\pi_1(X)$ = path components of ΩX require that quot top is discrete.)

Sketch pf: for any cover $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ have a map

$$\{\text{paths } \gamma : I \to X \mid \gamma(0) = x_0\} \Big/ \text{htpy rel endpoints} \longrightarrow \tilde{X}$$

$$[\gamma] \longmapsto \tilde{\gamma}(1)$$

e.g.

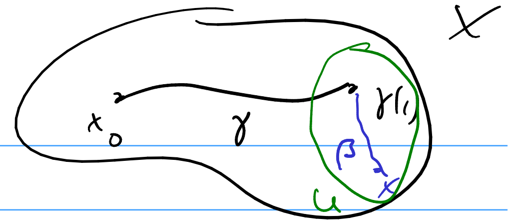


it's surjective. if $\pi_1(\tilde{X}) = 1$ then it's bijective

If we're given X but not $\tilde{X}$, define

$$\tilde{X} = \{\gamma : I \to X \mid \gamma(0) = x_0\} \Big/ \text{htpy rel endpoints}$$

topology on $\tilde{X}$: given $[\gamma] \in \tilde{X}$, let U be a simply conn. nbd. of $\gamma(1) \in X$



for any $x \in U$, choose a path β from $\gamma(1)$ to x.

bec. U is simply-conn, $[\beta]$ only depends on x

let $\tilde{U} = \{ [\gamma \cdot \beta] \mid \beta \text{ from points } x \in U \text{ as above} \} \subset \tilde{X}$

can check: these form the basis for a topology

p is cont. w/r/t this top. and it's a cover.

why is $\tilde{X}$ simply connected?

let e be the const path in X based at x_0

$[e] \in \tilde{X}$ will be our basepoint.

let $\alpha : I \to \tilde{X}$ be a path based at $[e]$

let $\beta = p \circ \alpha$ a path in X based at x_0.

for $t \in I$, define $\beta_t(s) = \beta(ts)$, a path in X from x_0 to $\beta(t)$

define $\tilde{\beta}(t) = \underbrace{[\beta_t]}_{\text{elt of } \tilde{X}}$, a path in $\tilde{X}$

$\tilde{\beta}$ lifts β and starts at $[e]$, hence $\tilde{\beta} = \alpha$

So $\alpha(1) = \tilde{\beta}(1)$ so $[e] = [\beta] \in \pi_1(X, x_0)$

i.e. $e \simeq \beta$

so $p_*[\alpha] = [\beta] = [e] \in \pi_1(X, x_0)$

but p_* is injective so $[\alpha] = 1 \in \pi_1(\tilde{X}, [e])$

$[\alpha]$ was arbitrary, so $\pi_1(\tilde{X}) = 1$. $\boxed{}$